

On the Capacity Region and the Generalized Degrees of Freedom Region for MIMO Interference Channel with Feedback

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Abstract

In this paper, we study the effect of feedback on two-user MIMO interference channels. The capacity region of MIMO interference channels with feedback is characterized within a constant number of bits, where this constant is independent of the channel matrices. Further, it is shown that the capacity region of a MIMO interference channel with feedback and its reciprocal interference channel are within a constant number of bits. Finally, the generalized degrees of freedom region for the MIMO interference channel with feedback is characterized.

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I. INTRODUCTION

Wireless networks with multiple users are interference-limited rather than noise-limited. Interference channel (IC) is a good starting point for understanding the performance limits of the interference limited communications [1–7]. Feedback can be employed in ICs to achieve an improvement in data rates [8–13]. However, most of the existing works on ICs with feedback are limited to discrete memoryless channels or to single-input single-output (SISO) channels. This paper analyzes multiple-input multiple-output (MIMO) Gaussian interference channels with feedback. In particular, we focus on a two-user MIMO IC with feedback.

The authors of [8] considered a SISO Gaussian IC with feedback and found its capacity region within two bits. It was shown that the capacity regions of Gaussian ICs increase unboundedly with feedback unlike the Gaussian multiple-access channel where the gains are bounded [14]. The degrees of freedom for a symmetric SISO Gaussian IC with feedback is also found in [8]. In this paper, we find an outer bound and an inner bound for the capacity region that differ by a constant number of bits, and also evaluate the generalized degrees of freedom (GDoF) region for a general MIMO IC with feedback.

The first main result of the paper is the characterization of the capacity region of a MIMO IC with feedback within $N_1 + N_2 + \max(N_1, N_2)$ bits, where N_1 and N_2 are the numbers of receive antennas at the two receivers. An outer-bound is obtained by first outer bounding the covariance matrices of both input signals and representing the outer bound as a region in terms of the cross-covariance matrix between the two input signals. This is further outer-bounded by a larger region that does not involve the cross-covariance matrix. The achievability strategy is based on block Markov encoding, backward decoding, and Han-Kobayashi message-splitting. This achievable rate and the outer bound are within $N_1 + N_2 + \max(N_1, N_2)$ bits of each other thus characterizing the capacity region of the two-user IC within constant number of bits where the constant is independent of the channel matrices. The achievability scheme that is used to prove the constant gap result assumes that the transmitted signals from the two transmitters in a time-slot are uncorrelated, unlike [8] where the signals were assumed correlated in the achievability. Thus, our achievable rate region is within 3 bits rather than 2 bits as in [8] of the capacity region of a SISO IC with feedback. An achievability scheme without correlated inputs was also shown to achieve within constant gap of the capacity region in [12] for a SISO IC with feedback. However, our gap between the inner and the outer bounds is smaller as compared to [12].

We note that the achievability strategies for a SISO IC in [8, 12] emphasize that the private part from

a transmitter using the Han-Kobayashi message splitting is such that it is received at the other receiver at the noise floor. However for a MIMO IC with feedback, it is not clear what its counterpart would be. The Han-Kobayashi message splitting used in this paper gives the notion of receiving the signal at the noise floor for a MIMO IC with feedback. Many matrix based results are derived in this paper to show a constant gap between the outer and the inner bounds of the capacity region of a MIMO IC with feedback, which may be of independent interest.

The second main result of the paper is to show that the capacity region of a MIMO IC with feedback and that of its corresponding reciprocal channel are within constant number of bits of each other, where the constant is independent of channel matrices. The reciprocal IC was considered in [4], where the authors showed that the capacity region of a MIMO IC without feedback is within constant number of bits of its corresponding reciprocal IC. This paper shows that the constant gap between a MIMO IC and its reciprocal channel also holds in the presence of feedback.

The third main result of the paper is a complete characterization of the GDoF region of a general MIMO IC with feedback when the average signal quality of each link, say ρ_{ij} for link from transmitter i to receiver j , varies with a base signal-to-noise ratio (SNR) parameter, say SNR, as $\lim_{\text{SNR} \rightarrow \infty} \frac{\log \rho_{ij}}{\log \text{SNR}} = \alpha_{ij}$, where α_{ij} can be different for each link with $i, j \in \{1, 2\}$. In other words, the average link quality of each link can potentially have different exponents of a base SNR. As a special case, we consider a symmetric IC where the number of antennas at both transmitters is the same, the number of antennas at both receivers is the same, and the SNRs for the direct links and the cross links are SNR and SNR^α , $\alpha \geq 0$, respectively. We find the GDoF (the maximum symmetric point in the GDoF region) for a given α and show that the GDoF is a “V”-curve rather than a “W”-curve corresponding to the GDoF without feedback as in [5]. Similar result was obtained for a SISO IC in [8] while this paper extends it to a MIMO system.

The reminder of the paper is organized as follows. Section II introduces the model for a MIMO IC model with feedback, reciprocal IC and the GDoF region. Sections III and IV describe our results on the capacity region and the GDoF region respectively. Section V concludes the paper. The detailed proofs of various results are given in Appendices A-E.

II. CHANNEL MODEL AND PRELIMINARIES

In this section, we describe the channel model considered in this paper. A two-user MIMO IC consists of two transmitters and two receivers. Transmitter i is labeled as T_i and receiver j is labeled as D_j for $i, j \in \{1, 2\}$. Further, we assume T_i has M_i antennas and D_i has N_i antennas, $i \in \{1, 2\}$. Henceforth,

such a MIMO IC will be referred to as the (M_1, N_1, M_2, N_2) MIMO IC. We assume that the channel matrix between transmitter T_i and receiver D_j is denoted by $H_{ij} \in \mathbb{C}^{N_j \times M_i}$, for $i, j \in \{1, 2\}$. We shall consider a time-invariant or fixed channel where the channel matrices remain fixed for the entire duration of communication. We also incorporate a non-negative power attenuation factor, denoted as ρ_{ij} , for the signal transmitted from T_i to D_j . At time-instant t , transmitter T_i chooses a vector $X_i(t) \in \mathbb{C}^{M_i \times 1}$ and transmits $\sqrt{P_i}X_i(t)$ over the channel, where P_i is the average transmit power at transmitter T_i , and

$$\text{tr}(Q_{ii}) \leq 1, \quad (1)$$

where $Q_{ij} = \mathbb{E}(X_i X_j^\dagger)$ for $i, j \in \{1, 2\}$.

In this paper, we denote the conjugate transpose of the matrix S as S^\dagger . Further, $A \preceq B$ means $B - A$ is a positive semi-definite (p.s.d.) matrix and $A \succeq B$ if $B \preceq A$. The identity matrix of size $s \times s$ is denoted by I_s . Further, we define $x^+ \triangleq \max(x, 0)$.

By definition of Q_{ij} , we see that $Q_{ij} = Q_{ji}^\dagger$. We will sometimes denote $Q = Q_{12}$ when it does not lead to confusion.

In addition, since Q_{ii} is p.s.d., its eigenvalues are positive. $\text{tr}(Q_{ii})$ is sum of eigenvalues and (1) implies that it is less than or equal to 1. So, all eigenvalues of Q_{ii} are less than or equal to 1. According to Theorem 7.7.3. of [15] we get $Q_{ii} \preceq I$. Moreover, we have $0 \preceq Q_{ij}Q_{ij}^\dagger \preceq I$, where $0 \preceq Q_{ij}Q_{ij}^\dagger$ results from the fact that every matrix in the form of AA^\dagger is p.s.d. and $Q_{ij}Q_{ij}^\dagger \preceq I$ results from $\text{tr}(Q_{ij}Q_{ij}^\dagger) = \text{tr}(Q_{ii})\text{tr}(Q_{jj}) \leq 1$ which gives $Q_{ij}Q_{ij}^\dagger \preceq I$ with a similar argument as we had for Q_{ii} .

The received signal at receiver D_i at time instant t is denoted as $Y_i(t)$ for $i \in \{1, 2\}$, and can be written as

$$Y_1(t) = \sqrt{\rho_{11}}H_{11}X_1(t) + \sqrt{\rho_{21}}H_{21}X_2(t) + Z_1(t), \quad (2)$$

$$Y_2(t) = \sqrt{\rho_{12}}H_{12}X_1(t) + \sqrt{\rho_{22}}H_{22}X_2(t) + Z_2(t), \quad (3)$$

where $Z_i(t) \in \mathbb{C}^{N_i \times 1}$ is independent and identically distributed (i.i.d.) $\text{CN}(0, I_{N_i})$ (complex Gaussian noise), ρ_{ii} is the received SNR at receiver D_i and ρ_{ij} is the received interference-to-noise-ratio at receiver D_j for $i, j \in \{1, 2\}$, $i \neq j$. A MIMO IC is fully described by three parameters. The first is the number of antennas at each transmitter and receiver, namely (M_1, N_1, M_2, N_2) . The second is the set of channel gains, $\overline{H} = \{H_{11}, H_{12}, H_{21}, H_{22}\}$. The third is the set of average link qualities of all the channels, $\overline{\rho} = \{\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}\}$. We assume that these parameters are known to all transmitters and receivers.

For MIMO IC with feedback, the transmitted signal at T_i , $X_i(t)$, is a function of the message W_i and the previous channel outputs at receiver D_i for $i \in \{1, 2\}$. Thus, the encoding functions of the two transmitters are given as

$$X_i(t) = f_i(W_i, Y_i^{t-1}), \quad i \in \{1, 2\}, \quad (4)$$

where f_{it} is the encoding function of T_i , W_i is the message of T_i and $Y_i^{t-1} = (Y_i(1), \dots, Y_i(t-1))$. Let us assume that T_i transmits information at a rate of R_i to receiver D_i using the codebook $C_{i,n}$ of length- n codewords with $|C_{i,n}| = 2^{nR_i}$. Given a message $m_i \in \{1, \dots, 2^{nR_i}\}$, the corresponding codeword $X_i^n(m_i) \in C_{i,n}$ satisfies the power constraint mentioned before. From the received signal Y_i^n , the receiver obtains an estimate \widehat{m}_i of the transmitted message m_i using a decoding function. Let the average probability of error be denoted by $e_{i,n} = \Pr(\widehat{m}_i \neq m_i)$.

A rate pair (R_1, R_2) is achievable if there exists a family of codebooks $C_{i,n}, i = \{1, 2\}_n$ and decoding functions such that $\max_i \{e_{i,n}\}$ goes to zero as the block length n goes to infinity. The capacity region $C(\overline{H}, \overline{\rho})$ of the IC with parameters \overline{H} and $\overline{\rho}$ is defined as the closure of the set of all achievable rate pairs.

Consider a two-dimensional rate region \mathcal{C} . Then, the region $\mathcal{C} \oplus ([0, a] \times [0, b])$ denotes the region formed by $\{(R_1, R_2) : R_1, R_2 \geq 0, ((R_1 - a)^+, (R_2 - b)^+) \in \mathcal{C}\}$ for some $a, b \geq 0$. Similarly, the region $\mathcal{C} \ominus ([0, a] \times [0, b])$ denotes the region formed by $\{(R_1, R_2) : R_1, R_2 \geq 0, ((R_1 + a)^+, (R_2 + b)^+) \in \mathcal{C}\}$ for some $a, b \geq 0$. Further, we define the notion of an achievable rate region that is within a constant number of bits of the capacity region as follows.

Definition 1. An achievable rate region $A(\overline{H}, \overline{\rho})$ is said to be within b bits of the capacity region if $A(\overline{H}, \overline{\rho}) \subseteq C(\overline{H}, \overline{\rho})$ and $A(\overline{H}, \overline{\rho}) \oplus ([0, b] \oplus [0, b]) \supseteq C(\overline{H}, \overline{\rho})$.

In this paper, we will use the GDoF region to characterize the capacity region of the MIMO IC with feedback in the limit of high SNR. This notion generalizes the conventional degrees of freedom (DoF) region metric by additionally emphasizing the signal level as a signaling dimension. It characterizes the simultaneously accessible fractions of spatial and signal-level dimensions (per channel use) by the two users when all the average channel coefficients vary as exponents of a nominal SNR parameter. Thus, we assume that

$$\lim_{\log(\text{SNR}) \rightarrow \infty} \frac{\log(\rho_{ij})}{\log(\text{SNR})} = \alpha_{ij}, \quad (5)$$

where $\alpha_{ij} \in \mathbb{R}^+$ for all $i, j \in \{1, 2\}$. In the limit of high SNR, the capacity region diverges.

The GDoF region is defined as the region formed by the set of all (d_1, d_2) such that $(d_1 \log(\text{SNR}) - o(\log(\text{SNR})), d_2 \log(\text{SNR}) - o(\log(\text{SNR})))^1$ is inside the capacity region. Thus, the GDoF is a function of link quality scaling exponents α_{ij} . We note that since the channel matrices are of full ranks with probability 1, we will have the GDoF with probability 1 over the randomness of channel matrices.

The property of maintaining the same performance even if the direction of information flow is reversed is known as the *reciprocity* of the channel. For a MIMO IC with parameters (M_1, N_1, M_2, N_2) , $\bar{H} = \{H_{11}, H_{12}, H_{21}, H_{22}\}$, and $\bar{\rho} = \{\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}\}$, the reciprocal MIMO IC has parameters (N_1, M_1, N_2, M_2) , $\bar{H}^R = \{H_{11}^T, H_{21}^T, H_{12}^T, H_{22}^T\}$, and $\bar{\rho}^R = \{\rho_{11}, \rho_{21}, \rho_{12}, \rho_{22}\}$.

III. CAPACITY REGION OF MIMO INTERFERENCE CHANNEL WITH FEEDBACK

In this section, we will describe our results on the capacity region of the two-user MIMO IC with feedback.

Our first result gives an outer bound on the capacity region of the two-user MIMO IC with feedback. Let $\mathcal{R}_o(Q)$ be the region formed by (R_1, R_2) satisfying the following constraints for some covariance matrix $Q = \mathbb{E}[X_1 X_2^\dagger]$:

¹ $a = o(\log(\text{SNR}))$ indicates that $\lim_{\text{SNR} \rightarrow \infty} \frac{a}{\log(\text{SNR})} = 0$.

$$R_1 \leq \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger + \sqrt{\rho_{11}\rho_{21}}H_{11}QH_{21}^\dagger + \sqrt{\rho_{11}\rho_{21}}H_{21}Q^\dagger H_{11}^\dagger), \quad (6)$$

$$R_2 \leq \log \det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger + \sqrt{\rho_{22}\rho_{12}}H_{22}Q^\dagger H_{12}^\dagger + \sqrt{\rho_{22}\rho_{12}}H_{12}QH_{22}^\dagger), \quad (7)$$

$$R_1 \leq \log \det \left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger - \rho_{12}H_{12}QQ^\dagger H_{12}^\dagger \right) + \log \det \left(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \right. \\ \left. \begin{bmatrix} \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^\dagger & \sqrt{\rho_{11}}H_{11}Q \end{bmatrix} \begin{bmatrix} I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger & \sqrt{\rho_{12}}H_{12}Q \\ \sqrt{\rho_{12}}Q^\dagger H_{12}^\dagger & I_{M_2} \end{bmatrix}^{-1} \right. \\ \left. \begin{bmatrix} \sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^\dagger \\ \sqrt{\rho_{11}}Q^\dagger H_{11}^\dagger \end{bmatrix} \right), \quad (8)$$

$$R_2 \leq \log \det \left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger - \rho_{21}H_{21}Q^\dagger QH_{21}^\dagger \right) + \log \det \left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger - \right. \\ \left. \begin{bmatrix} \sqrt{\rho_{22}\rho_{21}}H_{22}H_{21}^\dagger & \sqrt{\rho_{22}}H_{22}Q^\dagger \end{bmatrix} \begin{bmatrix} I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger & \sqrt{\rho_{21}}H_{21}Q^\dagger \\ \sqrt{\rho_{21}}QH_{21}^\dagger & I_{M_1} \end{bmatrix}^{-1} \right. \\ \left. \begin{bmatrix} \sqrt{\rho_{22}\rho_{21}}H_{21}H_{22}^\dagger \\ \sqrt{\rho_{22}}QH_{22}^\dagger \end{bmatrix} \right), \quad (9)$$

$$R_1 + R_2 \leq \log \det \left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger + \sqrt{\rho_{22}\rho_{12}}H_{22}Q^\dagger H_{12}^\dagger + \sqrt{\rho_{22}\rho_{12}}H_{12}QH_{22}^\dagger \right) \\ + \log \det \left(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \begin{bmatrix} \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^\dagger & \sqrt{\rho_{11}}H_{11}Q \end{bmatrix} \right. \\ \left. \begin{bmatrix} I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger & \sqrt{\rho_{12}}H_{12}Q \\ \sqrt{\rho_{12}}Q^\dagger H_{12}^\dagger & I_{M_2} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^\dagger \\ \sqrt{\rho_{11}}Q^\dagger H_{11}^\dagger \end{bmatrix} \right), \quad (10)$$

$$R_1 + R_2 \leq \log \det \left(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger + \sqrt{\rho_{11}\rho_{21}}H_{11}QH_{21}^\dagger + \sqrt{\rho_{11}\rho_{21}}H_{21}Q^\dagger H_{11}^\dagger \right) \\ + \log \det \left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger - \begin{bmatrix} \sqrt{\rho_{22}\rho_{21}}H_{22}H_{21}^\dagger & \sqrt{\rho_{22}}H_{22}Q^\dagger \end{bmatrix} \right. \\ \left. \begin{bmatrix} I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger & \sqrt{\rho_{21}}H_{21}Q^\dagger \\ \sqrt{\rho_{21}}QH_{21}^\dagger & I_{M_1} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{22}\rho_{21}}H_{21}H_{22}^\dagger \\ \sqrt{\rho_{22}}QH_{22}^\dagger \end{bmatrix} \right). \quad (11)$$

Further, let \mathcal{R}_o be the convex hull of $\mathcal{R}_o(Q)$ for all covariance matrices Q . The following theorem outer bounds the capacity region of the two-user MIMO IC with feedback.

Theorem 1. *The capacity region of the two-user MIMO IC with perfect feedback \mathcal{C}_{FB} is bounded from*

above as follows

$$\mathcal{C}_{FB} \subseteq \mathcal{R}_o. \quad (12)$$

Proof: The proof is given in Appendix A. ■

From the definition of $\mathcal{R}_o(Q)$, by substituting $Q = 0$ and after some simplifications, we get that $\mathcal{R}_o(0)$ is the region formed by (R_1, R_2) satisfying the following

$$R_1 \leq \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger), \quad (13)$$

$$R_2 \leq \log \det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger), \quad (14)$$

$$R_1 \leq \log \det \left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger \right) + \log \det \left(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^\dagger(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}\sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^\dagger \right), \quad (15)$$

$$R_2 \leq \log \det \left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger \right) + \log \det \left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger - \sqrt{\rho_{22}\rho_{21}}H_{22}H_{21}^\dagger(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger)^{-1}\sqrt{\rho_{22}\rho_{21}}H_{21}H_{22}^\dagger \right), \quad (16)$$

$$R_1 + R_2 \leq \log \det \left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger \right) + \log \det \left(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^\dagger(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}\sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^\dagger \right), \quad (17)$$

$$R_1 + R_2 \leq \log \det \left(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger \right) + \log \det \left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger - \sqrt{\rho_{22}\rho_{21}}H_{22}H_{21}^\dagger(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger)^{-1}\sqrt{\rho_{22}\rho_{21}}H_{21}H_{22}^\dagger \right). \quad (18)$$

The following result gives an inner bound to the capacity region of the two-user MIMO IC with feedback.

Theorem 2. *The capacity region for the two-user MIMO IC with perfect feedback \mathcal{C}_{FB} is bounded from below as*

$$\mathcal{C}_{FB} \supseteq \mathcal{R}_o(0) \ominus ([0, N_1 + N_2] \times [0, N_1 + N_2]). \quad (19)$$

Proof: The proof is provided in Appendix B. ■

The inner bound uses the achievable region for a two-user discrete memoryless IC with feedback as in [8]. The achievability scheme employs block Markov encoding, backward decoding, and Han-Kobayashi message-splitting. This result for a discrete memoryless channel is extended to MIMO IC with feedback

using a specific message splitting by power allocation. The transmitted signal from T_i , X_i , is given as

$$X_i = X_{ip} + X_{iu}, \quad (20)$$

where X_{ip} indicate the private message of T_i , and X_{iu} is the public message of T_i . We assume that $(X_{ip}$ and X_{iu} pairs are independent for $i = 1, 2$. However, these transmitted signals are correlated over time due to block Markov encoding. The private signal X_{ip} is chosen to be $X_{ip} \sim \text{CN}(0, K_{X_{ip}})$, and the public signal X_{iu} is chosen to be $X_{iu} \sim \text{CN}(0, K_{X_{iu}})$, where

$$K_{X_{ip}} = I_{M_i} - \sqrt{\rho_{ij}} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} \sqrt{\rho_{ij}} H_{ij}, \quad (21)$$

and

$$K_{X_{iu}} = I_{M_i} - K_{X_{ip}}, \quad (22)$$

for $i \in \{1, 2\}$.

We will show in Appendix B that the power allocation is feasible by showing $K_{X_{ip}} \succeq 0$ and $K_{X_{iu}} \succeq 0$. Further, this message split is such that the private signal is received at the other receiver with power bounded by a constant. More specifically, we have $\rho_{ji} H_{ji} K_{X_{ip}} H_{ji}^\dagger \preceq I_{N_j}$ thus showing that the effective received signal covariance matrix at receiver D_j corresponding to the private signal from transmitter T_i is at the noise floor.

This power allocation is different from that given in [8] even for a SISO channel. Note that the power split levels in the achievability scheme of [8] do not sum to 1 and thus do not satisfy the total power constraint. For the special case of SISO IC with feedback, the above gives a fix to the results in [8]. This power allocation assumes uncorrelated signals transmitted by the two users at each time-slot. The authors of [12] also used uncorrelated signals for SISO but had a larger gap between the inner and outer bounds for SISO IC with feedback than that achieved by our achievability strategy.

Having considered the inner and outer bounds for the capacity region of the two-user IC with feedback, the next result shows that the inner bound and the outer bound are within $N_1 + N_2 + \max(N_1, N_2)$ bits thus finding the capacity region of the two-user IC with feedback, approximately.

Theorem 3. *The capacity region for the two-user MIMO IC with perfect feedback C_{FB} is bounded from*

above and below as

$$\mathcal{R}_o(0) \ominus ([0, N_1 + N_2] \times [0, N_1 + N_2]) \subseteq \mathcal{C}_{FB} \subseteq \mathcal{R}_o(0) \oplus ([0, N_1] \times [0, N_2]), \quad (23)$$

where the inner and outer bounds are within $N_1 + N_2 + \max(N_1, N_2)$ bits.

Proof: The inner bound follows from Theorem 2. For outer bound, we outer-bound the region $\mathcal{R}_o(Q)$ as $\mathcal{R}_o(Q) \subseteq \mathcal{R}_o(0) \oplus ([0, N_1] \times [0, N_2])$ in Appendix C. Hence, $\mathcal{R}_o \subseteq \mathcal{R}_o(0) \oplus ([0, N_1] \times [0, N_2])$. Thus, using $Q = 0$ in $\mathcal{R}_o(Q)$ gives an approximate capacity region with the approximation gap as in the statement of the Theorem. ■

The authors of [8] found the capacity region for the SISO IC with feedback within 2 bits. The above Theorems generalize the result to find the capacity region of MIMO IC with feedback within $N_1 + N_2 + \max(N_1, N_2)$ bits. Note that the approximate capacity region without feedback in [4] involves bounds on $2R_1 + R_2$ which do not appear in our approximate capacity region with feedback. In addition, in [8], the approximate capacity region for the SISO IC with feedback involves the cross-covariance matrix of the inputs in the inner and outer bounds, whereas our approximate capacity region for the MIMO IC with feedback does not involve cross-covariance matrix of the inputs.

Figure 1 gives a pictorial representation for the result of Theorem 3. The inner and the outer bounds for the capacity region for MIMO IC with feedback are within a constant number of bits from the region $\mathcal{R}_o(0)$ and thus the inner and outer bound regions are within a constant number of bits of each other.

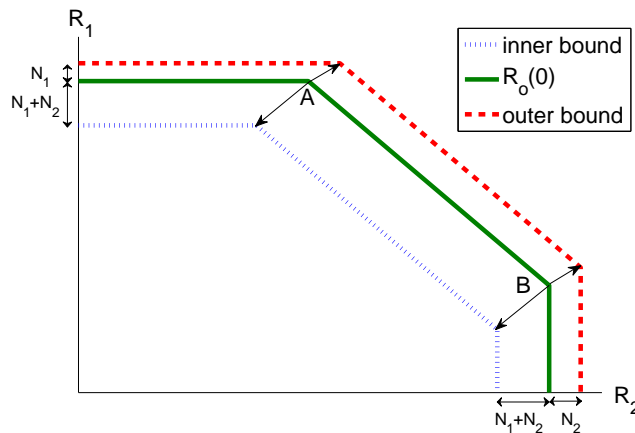


Fig. 1. Inner and outer bounds for the capacity region of MIMO IC with feedback are within a constant number of bits. The arrows from the corners A and B in $\mathcal{R}_o(0)$ toward their respective corners on outer bound have vertical length of N_1 and horizontal length of N_2 . The arrows from the corners A and B in $\mathcal{R}_o(0)$ toward their respective corners on inner bound have the vertical and horizontal length of $N_1 + N_2$ each.

Having characterized the approximate capacity region for the MIMO IC with feedback, we next explore

the relation of capacity region of the MIMO IC with feedback with that of the corresponding reciprocal MIMO IC with feedback. The next theorem shows that the capacity region of the MIMO IC with feedback is approximately the same as that of its corresponding reciprocal channel with feedback.

Theorem 4. *The capacity region for the two-user MIMO IC with feedback \mathcal{C}_{FB} and the capacity region for its corresponding reciprocal IC \mathcal{C}_{FB}^R with feedback are within constant gaps from each other. More precisely, the following expressions holds:*

$$\mathcal{R}_o(0) \ominus ([0, N_1 + N_2] \times [0, N_1 + N_2]) \subseteq \mathcal{C}_{FB} \subseteq \mathcal{R}_o(0) \oplus ([0, N_1] \times [0, N_2]), \quad (24)$$

$$\mathcal{R}_o(0) \ominus ([0, M_1 + M_2] \times [0, M_1 + M_2]) \subseteq \mathcal{C}_{FB}^R \subseteq \mathcal{R}_o(0) \oplus ([0, M_1] \times [0, M_2]). \quad (25)$$

Then, we get

$$\begin{aligned} \mathcal{C}_{FB}^R \ominus ([0, N_1 + N_2 + M_1] \times [0, N_1 + N_2 + M_2]) &\subseteq \mathcal{C}_{FB} \subseteq \\ \mathcal{C}_{FB}^R \oplus ([0, M_1 + M_2 + N_1] \times [0, M_1 + M_2 + N_2]), &\end{aligned} \quad (26)$$

$$\begin{aligned} \mathcal{C}_{FB} \ominus ([0, M_1 + M_2 + N_1] \times [0, M_1 + M_2 + N_2]) &\subseteq \mathcal{C}_{FB}^R \subseteq \\ \mathcal{C}_{FB} \oplus ([0, N_1 + N_2 + M_1] \times [0, N_1 + N_2 + M_2]). &\end{aligned} \quad (27)$$

Proof: In Appendix D, we show that the region $\mathcal{R}_o(0)$ for the MIMO IC is the same as the corresponding region $\mathcal{R}_o^R(0)$ for the corresponding reciprocal MIMO IC. Thus, (24)-(25) follow from Theorem 3. Moreover, (26)-(27) follow by simple manipulations on (24)-(25). ■

Thus, we see that the capacity region of a two-user MIMO IC with feedback and the corresponding reciprocal channel with feedback are within $N_1 + N_2 + M_1 + M_2 + \max(N_1 + M_1, N_2 + M_2)$ bits.

IV. GENERALIZED DEGREES OF FREEDOM REGION OF MIMO INTERFERENCE CHANNEL WITH FEEDBACK

This section describes our results on the GDoF region of the two-user MIMO IC with feedback. The GDoF gives the high SNR characterization of the capacity region. Since the inner and outer-bounds on the capacity region are within a constant gap, we characterize the exact GDoF region of the MIMO IC with feedback.

Define

$$f(u, (a_1, u_1), (a_2, u_2)) \triangleq \begin{cases} \min(u, u_1)a_1^+ + \min((u - u_1)^+, u_2)a_2^+, & \text{if } a_1 \geq a_2 \\ \min(u, u_2)a_2^+ + \min((u - u_2)^+, u_1)a_1^+, & \text{otherwise} \end{cases}. \quad (28)$$

The following result characterizes the GDoF for general MIMO IC with feedback for general power scaling parameters α_{ij} .

Theorem 5. *The GDoF region of the two-user MIMO IC with feedback is given by the set of (d_1, d_2) satisfying:*

$$\alpha_{11}d_1 \leq f(N_1, (\alpha_{11}, M_1), (\alpha_{21}, M_2)), \quad (29)$$

$$\alpha_{22}d_2 \leq f(N_2, (\alpha_{22}, M_2), (\alpha_{12}, M_1)), \quad (30)$$

$$\begin{aligned} \alpha_{11}d_1 \leq & \alpha_{12}\min(M_1, N_2) + \alpha_{11}\min((M_1 - N_2)^+, N_1) + \\ & (\alpha_{11} - \alpha_{12})^+(\min(M_1, N_1) - \min((M_1 - N_2)^+, N_1)), \end{aligned} \quad (31)$$

$$\begin{aligned} \alpha_{22}d_2 \leq & \alpha_{21}\min(M_2, N_1) + \alpha_{22}\min((M_2 - N_1)^+, N_2) + \\ & (\alpha_{22} - \alpha_{21})^+(\min(M_2, N_2) - \min((M_2 - N_1)^+, N_2)), \end{aligned} \quad (32)$$

$$\begin{aligned} \alpha_{11}d_1 + \alpha_{22}d_2 \leq & f(N_2, (\alpha_{22}, M_2), (\alpha_{12}, M_1)) + \alpha_{11}\min((M_1 - N_2)^+, N_1) + \\ & (\alpha_{11} - \alpha_{12})^+(\min(M_1, N_1) - \min((M_1 - N_2)^+, N_1)), \end{aligned} \quad (33)$$

$$\begin{aligned} \alpha_{11}d_1 + \alpha_{22}d_2 \leq & f(N_1, (\alpha_{11}, M_1), (\alpha_{21}, M_2)) + \alpha_{22}\min((M_2 - N_1)^+, N_2) + \\ & (\alpha_{22} - \alpha_{21})^+(\min(M_2, N_2) - \min((M_2 - N_1)^+, N_2)). \end{aligned} \quad (34)$$

Proof: According to Theorem 3, we can see that GDoF is equal to $\lim_{\text{SNR} \rightarrow \infty} \mathcal{R}_o(0)/\log \text{SNR}$. Appendix E evaluates the limit of $\mathcal{R}_o(0)/\log \text{SNR}$ as $\text{SNR} \rightarrow \infty$ to get the result as in the statement of the Theorem. ■

Since the capacity region of the MIMO IC with feedback and the corresponding reciprocal IC with feedback are within constant gap, the GDoF region of the MIMO IC with feedback and that of the corresponding reciprocal IC with feedback are the same, as given in the next corollary.

Corollary 1. *The GDoF region for the reciprocal IC with feedback is given by the set of (d_1, d_2) satisfying (29)-(34).*

We will now consider a special case of Theorem 5 where $M_1 = M_2 = M$, $N_1 = N_2 = N$, $\alpha_{11} = \alpha_{22} =$

1, and $\alpha_{12} = \alpha_{21} = \alpha$. This MIMO IC is called a symmetric MIMO IC. We also define GDoF d as the supremum over all d_i such that (d_i, d_i) is in the GDoF region. The GDoF for the symmetric MIMO IC with feedback is given as follows.

Corollary 2. *The GDoF for a two-user symmetric MIMO IC with feedback for $N \leq M$ is given as follows:*

$$GDoF_{PF} = \begin{cases} N - \frac{\alpha}{2}(2N - M)^+, & \text{if } \alpha \leq 1, \\ N(\frac{\alpha+1}{2}) - \frac{1}{2}(2N - M)^+, & \text{if } \alpha \geq 1. \end{cases} \quad (35)$$

Since the expressions are symmetric in N and M by Corollary 1, the GDoF for $M \leq N$ follow by interchanging the roles of M and N .

Proof: For the symmetric MIMO IC, we have

$$\begin{aligned} f(N_i, (\alpha_{ii}, M_i), (\alpha_{ji}, M_j)) &= f(N, (1, M), (\alpha, M)) \\ &= \max(1, \alpha) \min(M, N) + \min(1, \alpha) \min((N - M)^+, M). \end{aligned} \quad (36)$$

Let the GDoF be d . We will split the proof for $N \leq M$ in two cases.

Case 1 - $\alpha \leq 1$: We will go over all equations (29)-(34) and evaluate them for the symmetric case with $\alpha \leq 1$. Equations (29) and (30) can be simplified using (36) as follows

$$\begin{aligned} d &\leq \max(1, \alpha) \min(M, N) + \min(1, \alpha) \min((N - M)^+, M) \\ &= N. \end{aligned} \quad (37)$$

Equations (31) and (32) can be simplified as

$$\begin{aligned} d &\leq \alpha \min(M, N) + \min((M - N)^+, N) + (1 - \alpha)^+ (\min(M, N) - \min((M - N)^+, N)) \\ &= \alpha N + \min((M - N), N) + (1 - \alpha)N - (1 - \alpha) \min((M - N), N) \\ &= N + \alpha \min((M - N), N) \\ &= N + \alpha(N - (2N - M)^+). \end{aligned} \quad (38)$$

Equations (33) and (34) can be simplified as

$$\begin{aligned}
d &\leq \frac{1}{2}(\max(1, \alpha) \min(M, N) + \min(1, \alpha) \min((N - M)^+, M) + \min((M - N)^+, N) + \\
&\quad (1 - \alpha)^+(\min(M, N) - \min((M - N)^+, N))) \\
&= \frac{1}{2}(N + (1 - \alpha)N + \alpha \min((M - N), N)) \\
&= N - \frac{1}{2}\alpha(N - (N - (2N - M)^+)) \\
&= N - \frac{\alpha}{2}((2N - M)^+).
\end{aligned} \tag{39}$$

We note that the minimum of the right hand sides of (37), (38), and (39) would give us the GDoF. The minimum of these three terms is (39) which proves the result for $\alpha \leq 1$.

Case 2 - $\alpha \geq 1$:

In this case, equations (29) and (30) can be simplified as

$$\begin{aligned}
d &\leq \max(1, \alpha) \min(M, N) + \min(1, \alpha) \min((N - M)^+, M) \\
&= \alpha N.
\end{aligned} \tag{40}$$

Equations (31) and (32) can be simplified as

$$\begin{aligned}
d &\leq \alpha \min(M, N) + \min((M - N)^+, N) + \\
&\quad (1 - \alpha)^+(\min(M, N) - \min((M - N)^+, N)) \\
&= \alpha N + \min((M - N), N).
\end{aligned} \tag{41}$$

Equations (33) and (34) can be simplified as

$$\begin{aligned}
d &\leq \frac{1}{2}(\max(1, \alpha) \min(M, N) + \min(1, \alpha) \min((N - M)^+, M) + \min((M - N)^+, N) + \\
&\quad (1 - \alpha)^+(\min(M, N) - \min((M - N)^+, N))) \\
&= \frac{1}{2}(\alpha N + (N - (2N - M)^+)) \\
&= N \frac{(\alpha + 1)}{2} - \frac{1}{2}(2N - M)^+.
\end{aligned} \tag{42}$$

We note that the minimum of the right hand sides of (40), (41), and (42) would give us the GDoF. The minimum of these three terms is (42) which proves the result for $\alpha \geq 1$. ■

The authors of [5] found the GDoF for the two-user symmetric MIMO IC without feedback as follows

for $N \leq M$ (We can interchange the roles of N and M if $N > M$.)

$$GDoF_{NF} = \begin{cases} N - \alpha(2N - M)^+, & \text{if } 0 \leq \alpha \leq \frac{1}{2}, \\ N - (1 - \alpha)(2N - M)^+, & \text{if } \frac{1}{2} \leq \alpha \leq \frac{2}{3}, \\ N - \frac{\alpha}{2}(2N - M)^+, & \text{if } \frac{2}{3} \leq \alpha \leq 1, \\ \min\{N, N(\frac{\alpha+1}{2}) - \frac{1}{2}(2N - M)^+\}, & \text{if } \alpha \geq 1 \end{cases} \quad (43)$$

We note that the GDoF with and without feedback are the same for $\frac{2}{3} \leq \alpha \leq 1$. Figure 2 compares the GDoF for the two-user symmetric MIMO IC with and without feedback. In Figure 2(a), the “W”-curve obtained without feedback delineates the very weak ($0 \leq \alpha \leq \frac{1}{2}$), weak ($\frac{1}{2} \leq \alpha \leq \frac{2}{3}$), moderate ($\frac{2}{3} \leq \alpha \leq 1$), strong ($1 \leq \alpha \leq 3 - \frac{M}{N}$) and very strong ($3 - \frac{M}{N} \leq \alpha$) interference regimes. In the presence of feedback, the “W”-curve improves to a “V”-curve which delineates the weak ($0 \leq \alpha \leq 1$) and strong ($1 \leq \alpha$) interference regimes for all choices of N and M . For $\frac{M}{2} < N \leq M$, we see that the GDoF with feedback is strictly greater than that without feedback for $0 < \alpha < 2/3$ and for $\alpha > 3 - M/N$. For $N \leq M/2$, we see that the GDoF with feedback is strictly greater than that without feedback for $\alpha > 2$. The GDoF improvement indicates an unbounded gap in the corresponding capacity regions as the SNR goes to infinity.

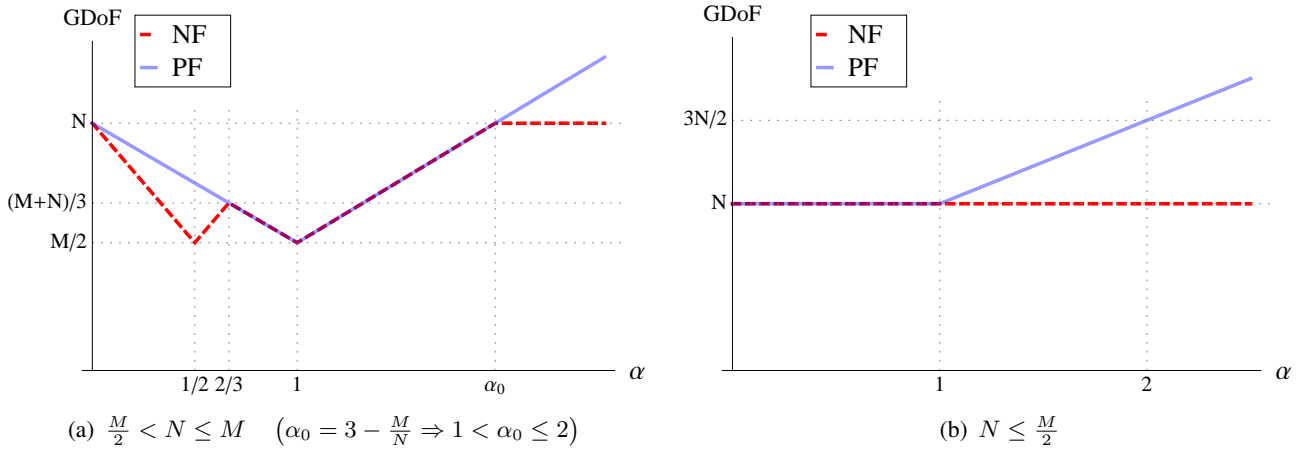


Fig. 2. GDoF for symmetric MIMO IC with perfect feedback (PF), and no-feedback (NF) for (a) $\frac{M}{2} < N \leq M$, and (b) $N \leq \frac{M}{2}$.

Interestingly, from Figure 2(b) we can see that if we increase M when $N \leq \frac{M}{2}$, the GDoF does not change. This can be interpreted as that while $N \leq \frac{M}{2}$, N act as a bottle-neck and increasing M does not increase the GDoF. As a special case consider a MISO IC for which we note that the GDoF is the same for all $M \geq 2$. Thus, increasing the transmit antennas beyond 2 does not increase the GDoF. However, increasing the transmit antennas from 1 to 2 gives a strict improvement in GDoF for all $\alpha > 0$. Similar

result also holds for SIMO systems where increasing the receive antennas from 1 to 2 help increase GDoF while increasing the receive antennas beyond 2 does not increase the GDoF.

V. CONCLUSIONS

This paper gives the capacity region of a MIMO IC with feedback within $N_1 + N_2 + \max(N_1, N_2)$ bits. The achievability is based on block Markov encoding, backward decoding, and Han-Kobayashi message-splitting. The capacity region for a MIMO IC with feedback is shown to be within a constant number of bits from the capacity region of the corresponding reciprocal IC. Further, the GDoF region for general MIMO IC is characterized. It is found that for symmetric IC, the GDoF form a “V”-curve rather than the “W”-curve without feedback.

APPENDIX A

PROOF OF OUTER BOUND FOR THEOREM 1

In this Appendix, we will show that $\mathcal{C}_{FB} \subseteq \mathcal{R}_o(Q)$ for some covariance matrix $Q = \mathbb{E}[X_1 X_2^\dagger]$.

The set of upper bounds to the capacity region will be derived in two steps. First, the capacity region is outer-bounded by a region defined in terms of the differential entropy of the random variables associated with the signals. These outer-bounds use genie-aided information at the receivers. Second, we outer-bound this region to prove the outer-bound as described in the statement of Theorem 1.

The following result outer-bounds the capacity region of two-user MIMO IC with feedback.

Lemma 1. *Let S_i be defined as $S_i \triangleq \sqrt{\rho_{ij}} H_{ij} X_i + Z_j$. Then, the capacity region of a two-user MIMO IC with feedback is outerbounded by the region formed by (R_1, R_2) satisfying*

$$R_1 \leq h(Y_1) - h(Z_1), \quad (44)$$

$$R_2 \leq h(Y_2) - h(Z_2), \quad (45)$$

$$R_1 \leq h(Y_2 | X_2) - h(Z_2) + h(Y_1 | X_2, S_1) - h(Z_1), \quad (46)$$

$$R_2 \leq h(Y_1 | X_1) - h(Z_1) + h(Y_2 | X_1, S_2) - h(Z_2), \quad (47)$$

$$R_1 + R_2 \leq h(Y_1 | S_1, X_2) - h(Z_2) + h(Y_2) - h(Z_1), \quad (48)$$

$$R_1 + R_2 \leq h(Y_2 | S_2, X_1) - h(Z_1) + h(Y_1) - h(Z_2). \quad (49)$$

Proof: The proof follows on the same lines as the proof of Theorem 3 in [8], replacing SISO channel gains by MIMO channel gains and is thus omitted in this paper. ■

The rest of the section outer-bounds this region to get the outer bound in Theorem 1. For this, we will introduce some useful Lemmas.

The next result outer-bounds the entropies and the conditional entropies of two random variables by their corresponding Gaussian random variables.

Lemma 2 ([16]). *Let X and Y be two random vectors, and let X^G and Y^G be Gaussian vectors with covariance matrices satisfying*

$$\text{Cov} \begin{bmatrix} X \\ Y \end{bmatrix} = \text{Cov} \begin{bmatrix} X^G \\ Y^G \end{bmatrix}, \quad (50)$$

Then, we have

$$h(Y) \leq h(Y^G), \quad (51)$$

$$h(Y | X) \leq h(Y^G | X^G). \quad (52)$$

The next result gives the determinant of a block matrix, which will be used extensively in the sequel.

Lemma 3 ([17]). *For block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with matrices A , B , C , and D , we have:*

$$\det M = \begin{cases} \det A \det(D - CA^{-1}B), & \text{if } A \text{ is invertible,} \\ \det D \det(A - BD^{-1}C), & \text{if } D \text{ is invertible.} \end{cases} \quad (53)$$

Now, we introduce a Lemma that is a key result which will be used to upper-bound a conditional entropy term in this section and also to show an upper bound in Appendix C.

Lemma 4. *Let $L(K, S)$ be defined as*

$$L(K, S) \triangleq K - KS(I_{N_2} + S^\dagger KS)^{-1}S^\dagger K, \quad (54)$$

for some $M_1 \times M_1$ p.s.d. Hermitian matrix K and some $M_1 \times N_2$ matrix S . Then if $0 \preceq K_1 \preceq K_2$ for

some Hermitian matrices K_1 and K_2 , we have

$$L(K_1, S) \preceq L(K_2, S). \quad (55)$$

Proof: We note that since K is p.s.d., $K + \epsilon I_{M_1}$ is invertible for all $\epsilon > 0$. Given $0 \preceq K_1 \preceq K_2$, let $F(\epsilon) \triangleq L(K_2 + \epsilon I_{M_1}, S) - L(K_1 + \epsilon I_{M_1}, S)$. We need to show that $F(0) \succeq 0$.

We first show that $F(\epsilon) \succeq 0$ for all $\epsilon > 0$. From Woodbury matrix identity (Appendix C.4.3 of [18]), we have that if A is invertible, $(A + BD)^{-1} = A^{-1} - A^{-1}B(I + DA^{-1}B)^{-1}DA^{-1}$. Thus, we have $L(K + \epsilon I_{M_1}, S) = ((K + \epsilon I_{M_1})^{-1} + SS^\dagger)^{-1}$ by substituting A as $(K + \epsilon I_{M_1})^{-1}$, B as S and D as S^\dagger in the above identity.

Thus, $F(\epsilon) = ((K_2 + \epsilon I_{M_1})^{-1} + SS^\dagger)^{-1} - ((K_1 + \epsilon I_{M_1})^{-1} + SS^\dagger)^{-1}$. Since K_1 and K_2 are Hermitian p.s.d. matrices with $K_1 \preceq K_2$, it easily follows that $F(\epsilon) \succeq 0$.

Having shown that $F(\epsilon) \succeq 0$ for all $\epsilon > 0$, we will now prove the continuity of $F(\epsilon)$ at $\epsilon = 0$. For this, we take the partial derivative of $F(\epsilon)$ at $\epsilon = 0$ and show that it is not unbounded thus proving that $F(\epsilon)$ is continuous at $\epsilon = 0$. Thus, we have

$$\begin{aligned} \frac{dF(\epsilon)}{d\epsilon} &= \frac{d}{d\epsilon} L(K_2 + \epsilon I_{M_1}, S) - \frac{d}{d\epsilon} L(K_1 + \epsilon I_{M_1}, S) \\ &= \frac{d}{d\epsilon} L(K_2 + \epsilon I_{M_1}, S) - \frac{d}{d\epsilon} L(K_1 + \epsilon I_{M_1}, S) \end{aligned} \quad (56)$$

Thus, it is enough to show that $\lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} L(K_i + \epsilon I_{M_1}, S)$ is bounded. Thus, we have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} L(K_i + \epsilon I_{M_1}, S) &= \lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} (K_i + \epsilon I_{M_1} - (K_i + \epsilon I_{M_1})S(I_{N_2} + S^\dagger(K_i + \epsilon I_{M_1})S)^{-1}S^\dagger(K_i + \epsilon I_{M_1})) \\ &= I_{M_1} - \lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} ((K_i + \epsilon I_{M_1})S(I_{N_2} + S^\dagger(K_i + \epsilon I_{M_1})S)^{-1}S^\dagger(K_i + \epsilon I_{M_1})) \\ &= I_{M_1} - S(I_{N_2} + S^\dagger K_i S)^{-1}S^\dagger K_i - K_i S(I_{N_2} + S^\dagger K_i S)^{-1}S^\dagger \\ &\quad + K_i S(I_{N_2} + S^\dagger K_i S)^{-1}S^\dagger S(I_{N_2} + S^\dagger K_i S)^{-1}S^\dagger K_i, \end{aligned} \quad (57)$$

which is bounded. Thus, $F(\epsilon)$ is continuous at $\epsilon = 0$. Further, since K_1 and K_2 are Hermitian, we see that $F(\epsilon)$ is Hermitian and thus normal. From the Wielandt-Hoffman theorem [19], we note that the \mathbb{L}_2 norm of the difference in eigen-values (ordered in a particular way) of two normal matrices is bounded by the Frobenium norm of the difference of the two matrices. This shows that since $F(\epsilon) \succeq 0$ and $F(\epsilon) - F(0) \rightarrow 0$ as $\epsilon \rightarrow 0$, we have that the eigen-values of $F(\epsilon)$ approach the eigen-values of $F(0)$ as $\epsilon \rightarrow 0$. Thus, all the eigen-values of $F(0)$ are non-negative which proves that $F(0)$ is positive semi-definite

thus proving the result. ■

The next three Lemmas outer-bounds entropy and conditional entropies of some random variables.

Lemma 5. *The entropy of the received signal at the i^{th} receiver, $h(Y_i)$, is outer-bounded as follows*

$$\begin{aligned} h(Y_i) \leq & \log \det \left(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ii} Q_{ij} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ji} Q_{ij}^\dagger H_{ii}^\dagger \right) \\ & + \log \det (2\pi e I_{N_i}), \end{aligned} \quad (58)$$

for $i, j \in \{1, 2\}$, $i \neq j$.

Proof:

$$\begin{aligned} h(Y_i) & \stackrel{(a)}{\leq} h(Y_i^G) \\ & = \log \det 2\pi e \mathbb{E}[|Y_i^G|^2] \\ & = \log \det 2\pi e \left(I_{N_i} + \rho_{ii} H_{ii} Q_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} Q_{jj} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ii} Q_{ij} H_{ji}^\dagger \right. \\ & \quad \left. + \sqrt{\rho_{ii} \rho_{ji}} H_{ji} Q_{ij}^\dagger H_{ii}^\dagger \right) \\ & \stackrel{(b)}{\leq} \log \det 2\pi e \left(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ii} Q_{ij} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ji} Q_{ij}^\dagger H_{ii}^\dagger \right) \\ & = \log \det \left(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ii} Q_{ij} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ji} Q_{ij}^\dagger H_{ii}^\dagger \right) \\ & \quad + \log \det (2\pi e I_{N_i}), \end{aligned} \quad (59)$$

where (a) follows from Lemma 2, and (b) follows from the fact that $\log \det(\cdot)$ is a monotonically increasing function on the cone of positive definite matrices and we have $Q_{ii} \preceq I_{M_i \times M_i}$ for $i \in \{1, 2\}$.

Taking $2\pi e$ out of the above determinant in the last part, gives the result as in the statement of the Lemma. ■

Lemma 6. *The conditional entropy of the received signal at the i^{th} receiver given the transmitted signal from the i^{th} transmitter, $h(Y_i|X_i)$ is outer-bounded as follows*

$$h(Y_i|X_i) \leq \log \det \left(I_{N_i} + \rho_{ji} H_{ji} H_{ji}^\dagger - \rho_{ji} H_{ji} Q_{ij}^\dagger Q_{ij} H_{ji}^\dagger \right) + \log \det 2\pi e (I_{N_i}), \quad (60)$$

where Q_{ij} is the cross-covariance between X_i and X_j and Q_{ii} is the covariance matrix for X_i .

Proof: Let

$$K_{i1} \triangleq \mathbb{E} \begin{bmatrix} X_i X_i^\dagger & X_i Y_i^\dagger \\ Y_i X_i^\dagger & Y_i Y_i^\dagger \end{bmatrix} = \begin{bmatrix} Q_{ii} & \sqrt{\rho_{ii}} Q_{ii} H_{ii}^\dagger + \sqrt{\rho_{ji}} Q_{ij} H_{ji}^\dagger \\ \sqrt{\rho_{ii}} H_{ii} Q_{ii} + \sqrt{\rho_{ji}} H_{ji} Q_{ij}^\dagger & \mathbb{E}[Y_i Y_i^\dagger] \end{bmatrix}. \quad (61)$$

where

$$\mathbb{E}[Y_i Y_i^\dagger] = I_{N_i} + \rho_{ii} H_{ii} Q_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} Q_{jj} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ii} Q_{ij} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ji} Q_{ij}^\dagger H_{ii}^\dagger, \quad (62)$$

and

$$K_{i2} \triangleq \mathbb{E}[X_i X_i^\dagger] = Q_{ii}. \quad (63)$$

According to Lemma 2, we get

$$\begin{aligned} h(Y_i | X_i) &\leq h(Y_i^G | X_i^G) \\ &= h(X_i^G, Y_i^G) - h(X_i^G) \\ &= \log \det 2\pi e(K_{i1}) - \log \det 2\pi e(K_{i2}) \\ &= \log \det(K_{i1}) - \log \det(K_{i2}) + \log \det 2\pi e(I_{N_i}). \end{aligned} \quad (64)$$

Due to the reason that Q 's elements are chosen from a continuous space, it is invertible with probability of one. In addition, according to Corollary 7.7.4(a) of [15], if we have $Q_{ii} \preceq I_{M_i \times M_i}$, $Q_{ii}^{-1} \succeq I_{M_i \times M_i}$. Using Lemma 3 with $M = K_{i1}$ and $A = K_{i2}$, we get

$$\begin{aligned} \log \det K_{i1} &= \log(\det(\mathbb{E}(X_i X_i^\dagger)) \det(\mathbb{E}(Y_i Y_i^\dagger) - \mathbb{E}(Y_i X_i^\dagger)(\mathbb{E}(X_i X_i^\dagger))^{-1} \mathbb{E}(X_i Y_i^\dagger))) \\ &= \log \det(\mathbb{E}(X_i X_i^\dagger)) + \log \det(\mathbb{E}(Y_i Y_i^\dagger) - \mathbb{E}(Y_i X_i^\dagger)(\mathbb{E}(X_i X_i^\dagger))^{-1} \mathbb{E}(X_i Y_i^\dagger)) \\ &\stackrel{(a)}{=} \log \det(Q_{ii}) + \log \det(I_{N_i} + \rho_{ji} H_{ji} Q_{jj} H_{ji}^\dagger - \rho_{ji} H_{ji} Q_{ij}^\dagger Q_{ii}^{-1} Q_{ij} H_{ji}^\dagger) \\ &\stackrel{(b)}{\leq} \log \det(Q_{ii}) + \log \det(I_{N_i} + \rho_{ji} H_{ji} H_{ji}^\dagger - \rho_{ji} H_{ji} Q_{ij}^\dagger Q_{ij} H_{ji}^\dagger), \end{aligned} \quad (65)$$

where (a) is obtained by using (61) and some simplifications, and (b) follows from the fact that $\log \det(\cdot)$ is a monotonically increasing function on the cone of positive definite matrices and we have $Q_{ii} \preceq I_{M_i \times M_i}$ and $Q_{ii}^{-1} \succeq I_{M_i \times M_i}$ according to Corollary 7.7.4(a) of [15] for $i, j \in \{1, 2\}$, $i \neq j$.

Substituting (65) in (64) gives the result as in the statement of the Lemma. \blacksquare

Lemma 7. *The conditional entropy of the received signal at the i^{th} receiver given X_j and S_i , $h(Y_i|X_j, S_i)$ is outer-bounded as follows*

$$\begin{aligned}
 h(Y_i | X_j, S_i) \leq & \log \det \left(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger - \begin{bmatrix} \sqrt{\rho_{ii} \rho_{ij}} H_{ii} H_{ij}^\dagger & \sqrt{\rho_{ii}} H_{ii} Q_{ij} \end{bmatrix} \right. \\
 & \left. \begin{bmatrix} I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger & \sqrt{\rho_{ij}} H_{ij} Q_{ij} \\ \sqrt{\rho_{ij}} Q_{ij}^\dagger H_{ij}^\dagger & I_{M_j} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{ii} \rho_{ij}} H_{ij} H_{ii}^\dagger \\ \sqrt{\rho_{ii}} Q_{ij}^\dagger H_{ii}^\dagger \end{bmatrix} \right) \\
 & + \log \det 2\pi e (I_{N_i}).
 \end{aligned} \tag{66}$$

Proof: Let K_{i3} and K_{i4} be defined as follows

$$\begin{aligned}
 K_{i3} &\triangleq \mathbb{E} \left[\begin{bmatrix} \sqrt{\rho_{ii}} H_{ii} X_i + Z_i \\ \sqrt{\rho_{ij}} H_{ij} X_i + Z_j \\ X_j \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\rho_{ii}} H_{ii} X_i + Z_i \\ \sqrt{\rho_{ij}} H_{ij} X_i + Z_j \\ X_j \end{bmatrix}^\dagger \right] \\
 &= \begin{bmatrix} I_{N_i} + \rho_{ii} H_{ii} Q_{ii} H_{ii}^\dagger & \sqrt{\rho_{ii} \rho_{ij}} H_{ii} Q_{ii} H_{ij}^\dagger & \sqrt{\rho_{ii}} H_{ii} Q_{ij} \\ \sqrt{\rho_{ii} \rho_{ij}} H_{ij} Q_{ii} H_{ii}^\dagger & I_{N_j} + \rho_{ij} H_{ij} Q_{ii} H_{ij}^\dagger & \sqrt{\rho_{ij}} H_{ij} Q_{ij} \\ \sqrt{\rho_{ii}} Q_{ij}^\dagger H_{ii}^\dagger & \sqrt{\rho_{ij}} Q_{ij}^\dagger H_{ij}^\dagger & Q_{jj} \end{bmatrix},
 \end{aligned} \tag{67}$$

and

$$\begin{aligned}
 K_{i4} &\triangleq \mathbb{E} \left[\begin{bmatrix} \sqrt{\rho_{ij}} H_{ij} X_i + Z_j \\ X_j \end{bmatrix} \begin{bmatrix} \sqrt{\rho_{12}} H_{ij} X_i + Z_j \\ X_j \end{bmatrix}^\dagger \right] \\
 &= \begin{bmatrix} I_{N_j} + \rho_{ij} H_{ij} Q_{ii} H_{ij}^\dagger & \sqrt{\rho_{ij}} H_{ij} Q_{ij} \\ \sqrt{\rho_{ij}} Q_{ij}^\dagger H_{ij}^\dagger & Q_{jj} \end{bmatrix}.
 \end{aligned} \tag{68}$$

Further, let $Y'_i = \sqrt{\rho_{ii}}H_{ii}X_i + Z_i$. Then,

$$\begin{aligned}
h(Y_i|X_j, S_i) &= h(\sqrt{\rho_{ii}}H_{ii}X_i + \sqrt{\rho_{ji}}H_{ji}X_j + Z_i \mid X_j, \sqrt{\rho_{ij}}H_{ij}X_i + Z_j) \\
&= h(\sqrt{\rho_{ii}}H_{ii}X_i + Z_i \mid X_j, \sqrt{\rho_{ij}}H_{ij}X_i + Z_j) \\
&= h(Y'_i|X_j, S_i) \\
&\stackrel{(a)}{\leq} h(Y_i'^G|S_i^G, X_j^G) \\
&= h(Y_i'^G, S_i^G, X_j^G) - h(S_i^G, X_j^G) \\
&= \log \det 2\pi e(K_{i3}) - \log \det 2\pi e(K_{i4}) \\
&= \log \det(K_{i3}) - \log \det(K_{i4}) + \log \det 2\pi e(I_{N_i}),
\end{aligned} \tag{69}$$

where (a) follows from Lemma 2 by taking the two vectors S_i and X_j of lengths N_j and M_j , respectively, together as a single vector of length of $N_j + M_j$ and then, used Lemma 2.

Substituting $M = K_{i3}$ and $D = K_{i4}$ in Lemma 3, we get

$$\begin{aligned}
\log \det(K_{i3}) &= \log \det(K_{i4}) + \log \det \left(I_{N_i} + \rho_{ii}H_{ii}Q_{ii}H_{ii}^\dagger \right. \\
&\quad \left. - \begin{bmatrix} \sqrt{\rho_{ii}\rho_{ij}}H_{ii}Q_{ii}H_{ij}^\dagger & \sqrt{\rho_{ii}}H_{ii}Q_{ij} \end{bmatrix} [(K_{i4})]^{-1} \begin{bmatrix} \sqrt{\rho_{ii}\rho_{ij}}H_{ij}Q_{ii}H_{ii}^\dagger \\ \sqrt{\rho_{ii}}Q_{ij}^\dagger H_{ii}^\dagger \end{bmatrix} \right) \\
&= \log \det(K_{i4}) + \log \det \left(I_{N_i} + \rho_{ii}H_{ii}Q_{ii}H_{ii}^\dagger - \begin{bmatrix} \sqrt{\rho_{ii}\rho_{ij}}H_{ii}Q_{ii}H_{ij}^\dagger & \sqrt{\rho_{ii}}H_{ii}Q_{ij} \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} I_{N_j} + \rho_{ij}H_{ij}Q_{ii}H_{ij}^\dagger & \sqrt{\rho_{ij}}H_{ij}Q_{ij} \\ \sqrt{\rho_{ij}}Q_{ij}^\dagger H_{ij}^\dagger & Q_{jj} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{ii}\rho_{ij}}H_{ij}Q_{ii}H_{ii}^\dagger \\ \sqrt{\rho_{ii}}Q_{ij}^\dagger H_{ii}^\dagger \end{bmatrix} \right).
\end{aligned} \tag{70}$$

Note that since $Q_{jj} \preceq I$, using Lemma 3 we can see that $Q_{jj} = I$ outer-bounds the determinant of

$$\begin{bmatrix} I_{N_j} + \rho_{ij}H_{ij}Q_{ii}H_{ij}^\dagger & \sqrt{\rho_{ij}}H_{ij}Q_{ij} \\ \sqrt{\rho_{ij}}Q_{ij}^\dagger H_{ij}^\dagger & Q_{jj} \end{bmatrix}.$$

Since $B \preceq I$ implies $ABA^\dagger \preceq AA^\dagger$, we have that $Q_{jj} = I$ outer-bounds the expression of the right

hand side of (70). Thus,

$$\begin{aligned} \log \det (K_{i3}) &\leq \log \det (K_{i4}) + \log \det \left(I_{N_i} + \rho_{ii} H_{ii} Q_{ii} H_{ii}^\dagger - \begin{bmatrix} \sqrt{\rho_{ii} \rho_{ij}} H_{ii} Q_{ii} H_{ij}^\dagger & \sqrt{\rho_{ii}} H_{ii} Q_{ij} \end{bmatrix} \right. \\ &\quad \left. \begin{bmatrix} I_{N_j} + \rho_{ij} H_{ij} Q_{ii} H_{ij}^\dagger & \sqrt{\rho_{ij}} H_{ij} Q_{ij} \\ \sqrt{\rho_{ij}} Q_{ij}^\dagger H_{ij}^\dagger & I_{M_j} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{ii} \rho_{ij}} H_{ij} Q_{ii} H_{ii}^\dagger \\ \sqrt{\rho_{ii}} Q_{ij}^\dagger H_{ii}^\dagger \end{bmatrix} \right). \end{aligned} \quad (71)$$

Next, we will show that $Q_{ii} = I$ maximizes (71).

Let us define $S \triangleq \sqrt{\rho_{ij}} H_{ij}^\dagger$, $W \triangleq Q_{ii} - Q_{ij} Q_{ij}^\dagger$, $E \triangleq (I_{N_j} + S^\dagger W S)^{-1}$ and

$$f(S, Q_{ii}) \triangleq Q_{ii} - \begin{bmatrix} Q_{ii} S & Q_{ij} \end{bmatrix} \begin{bmatrix} I_{N_j} + S^\dagger Q_{ii} S & S^\dagger Q_{ij} \\ Q_{ij}^\dagger S & I_{M_j} \end{bmatrix}^{-1} \begin{bmatrix} S^\dagger Q_{ii} \\ Q_{ij}^\dagger \end{bmatrix}. \quad (72)$$

We can check that

$$\begin{bmatrix} I_{N_j} + S^\dagger Q_{ii} S & S^\dagger Q_{ij} \\ Q_{ij}^\dagger S & I_{M_j} \end{bmatrix} \begin{bmatrix} E & -ES^\dagger Q_{ij} \\ -Q_{ij}^\dagger S E & I + Q_{ij}^\dagger S E S^\dagger Q_{ij} \end{bmatrix} = I_{M_j + N_j}. \quad (73)$$

Hence:

$$\begin{aligned} f(S, Q_{ii}) &= Q_{ii} - \begin{bmatrix} Q_{ii} S & Q_{ij} \end{bmatrix} \begin{bmatrix} I_{N_j} + S^\dagger Q_{ii} S & S^\dagger Q_{ij} \\ Q_{ij}^\dagger S & I_{M_j} \end{bmatrix}^{-1} \begin{bmatrix} S^\dagger Q_{ii} \\ Q_{ij}^\dagger \end{bmatrix} \\ &= Q_{ii} - \begin{bmatrix} Q_{ii} S & Q_{ij} \end{bmatrix} \begin{bmatrix} E & -ES^\dagger Q_{ij} \\ -Q_{ij}^\dagger S E & I + Q_{ij}^\dagger S E S^\dagger Q_{ij} \end{bmatrix} \begin{bmatrix} S^\dagger Q_{ii} \\ Q_{ij}^\dagger \end{bmatrix} \\ &= Q_{ii} - Q_{ii} S E S^\dagger Q_{ii} + Q_{ii} S E S^\dagger Q_{ij} Q_{ij}^\dagger + Q_{ij} Q_{ij}^\dagger S E S^\dagger Q_{ii} - Q_{ij} Q_{ij}^\dagger - \\ &\quad Q_{ij} Q_{ij}^\dagger S E S^\dagger Q_{ij} Q_{ij}^\dagger \\ &= Q_{ii} - Q_{ij} Q_{ij}^\dagger - (Q_{ii} - Q_{ij} Q_{ij}^\dagger) S E S^\dagger (Q_{ii} - Q_{ij} Q_{ij}^\dagger) \\ &= Q_{ii} - Q_{ij} Q_{ij}^\dagger - (Q_{ii} - Q_{ij} Q_{ij}^\dagger) S (I + S^\dagger (Q_{ii} - Q_{ij} Q_{ij}^\dagger) S)^{-1} S^\dagger (Q_{ii} - Q_{ij} Q_{ij}^\dagger) \\ &= W - W S (I + S^\dagger W S)^{-1} S^\dagger W. \end{aligned} \quad (74)$$

We know that $W = Q_{ii} - Q_{ij} Q_{ij}^\dagger \preceq I - Q_{ij} Q_{ij}^\dagger$. So, according to Lemma 4 with K_1 as $Q_{ii} - Q_{ij} Q_{ij}^\dagger$ and K_2 as $I - Q_{ij} Q_{ij}^\dagger$, we have $f(S, Q_{ii}) \preceq f(S, I_{M_i})$. Thus, we use this outer-bound by replacing Q_{ii}

by I to get

$$\begin{aligned}
& \log \det (K_{i3}) - \log \det (K_{i4}) \\
\leq & \log \det \left(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger - \begin{bmatrix} \sqrt{\rho_{ii} \rho_{ij}} H_{ii} H_{ij}^\dagger & \sqrt{\rho_{ii}} H_{ii} Q_{ij} \end{bmatrix} \right. \\
& \left. \begin{bmatrix} I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger & \sqrt{\rho_{ij}} H_{ij} Q_{ij} \\ \sqrt{\rho_{ij}} Q_{ij}^\dagger H_{ij}^\dagger & I_{M_j} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{ii} \rho_{ij}} H_{ij} H_{ii}^\dagger \\ \sqrt{\rho_{ii}} Q_{ij}^\dagger H_{ii}^\dagger \end{bmatrix} \right). \tag{75}
\end{aligned}$$

Substituting this in (69), we get

$$\begin{aligned}
h(Y_i | X_j, S_i) & \leq \log \det (K_{i3}) - \log \det (K_{i4}) + \log \det 2\pi e (I_{N_i}) \\
& \leq \log \det \left(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger - \begin{bmatrix} \sqrt{\rho_{ii} \rho_{ij}} H_{ii} H_{ij}^\dagger & \sqrt{\rho_{ii}} H_{ii} Q_{ij} \end{bmatrix} \right. \\
& \quad \left. \begin{bmatrix} I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger & \sqrt{\rho_{ij}} H_{ij} Q_{ij} \\ \sqrt{\rho_{ij}} Q_{ij}^\dagger H_{ij}^\dagger & I_{M_j} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{ii} \rho_{ij}} H_{ij} H_{ii}^\dagger \\ \sqrt{\rho_{ii}} Q_{ij}^\dagger H_{ii}^\dagger \end{bmatrix} \right) \\
& \quad + \log \det 2\pi e (I_{N_i}). \tag{76}
\end{aligned}$$

■

The rest of the section considers the 6 terms in Lemma 1 and outer-bounds each of them to get the terms in the outer-bound of Theorem 1.

First term: For the first term in Lemma 1,

$$\begin{aligned}
R_1 & \leq h(Y_1) - h(Z_1) \\
& \stackrel{(a)}{\leq} \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger + \rho_{21} H_{21} H_{21}^\dagger + \sqrt{\rho_{11} \rho_{21}} H_{11} Q_{12} H_{21}^\dagger + \sqrt{\rho_{11} \rho_{21}} H_{21} Q_{12}^\dagger H_{11}^\dagger \right) \\
& \quad + \log \det 2\pi e (I_{N_1}) - h(Z_1) \\
& \stackrel{(b)}{=} \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger + \rho_{21} H_{21} H_{21}^\dagger + \sqrt{\rho_{11} \rho_{21}} H_{11} Q_{12} H_{21}^\dagger + \sqrt{\rho_{11} \rho_{21}} H_{21} Q_{12}^\dagger H_{11}^\dagger \right), \tag{77}
\end{aligned}$$

where (a) follows from Lemma 5 and (b) follows from the fact that $h(Z_1) = \log \det 2\pi e (I_{N_1})$.

Second term: The second bound is similar to the first bound by replacing 1 and 2 in the indices.

Third term: For the third bound in Lemma 1, it is sufficient to replace upper bounds of $h(Y_2 | X_2)$ and

$h(Y_1|X_2, S_1)$ from Lemma 6 and Lemma 7 as follows

$$\begin{aligned}
R_1 &\leq h(Y_2 | X_2) - h(Z_2) + h(Y_1|X_2, S_1) - h(Z_1) \\
&\stackrel{(a)}{\leq} \log \det \left(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger - \rho_{12} H_{12} Q_{21}^\dagger Q_{21} H_{12}^\dagger \right) + \log \det 2\pi e (I_{N_2}) \\
&\quad + \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \begin{bmatrix} \sqrt{\rho_{11}\rho_{12}} H_{11} H_{12}^\dagger & \sqrt{\rho_{11}} H_{11} Q_{12} \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger & \sqrt{\rho_{12}} H_{12} Q_{12} \\ \sqrt{\rho_{12}} Q_{12}^\dagger H_{12}^\dagger & I_{M_2} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{11}\rho_{12}} H_{12} H_{11}^\dagger \\ \sqrt{\rho_{11}} Q_{12}^\dagger H_{11}^\dagger \end{bmatrix} \right) \\
&\quad + \log \det 2\pi e (I_{N_1}) - h(Z_1) - h(Z_2) \\
&\stackrel{(b)}{=} \log \det \left(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger - \rho_{12} H_{12} Q_{21}^\dagger Q_{21} H_{12}^\dagger \right) \\
&\quad + \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \begin{bmatrix} \sqrt{\rho_{11}\rho_{12}} H_{11} H_{12}^\dagger & \sqrt{\rho_{11}} H_{11} Q_{12} \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger & \sqrt{\rho_{12}} H_{12} Q_{12} \\ \sqrt{\rho_{12}} Q_{12}^\dagger H_{12}^\dagger & I_{M_2} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{11}\rho_{12}} H_{12} H_{11}^\dagger \\ \sqrt{\rho_{11}} Q_{12}^\dagger H_{11}^\dagger \end{bmatrix} \right), \tag{78}
\end{aligned}$$

where (a) is obtained by using Lemma 6 and Lemma 7 and (b) follows from the fact that $h(Z_i) = \log \det 2\pi e (I_{N_i})$, for $i = 1, 2$.

Fourth term: The fourth term is similar to the third term by replacing 1 and 2 in the indices.

Fifth term: According to the fifth bound in Lemma 1, it is sufficient to replace upper bounds of $h(Y_1|X_2, S_1)$ and $h(Y_2)$ from Lemma 7 and Lemma 5, respectively, and get the fifth bound of

Theorem 1 as follows

$$\begin{aligned}
R_1 + R_2 &\leq h(Y_1 | S_1, X_2) - h(Z_2) + h(Y_2) - h(Z_1) \\
&\stackrel{(a)}{\leq} \log \det \left(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger + \rho_{12} H_{12} H_{12}^\dagger + \sqrt{\rho_{22} \rho_{12}} H_{22} Q_{12} H_{12}^\dagger + \sqrt{\rho_{22} \rho_{12}} H_{12} Q_{12}^\dagger H_{22}^\dagger \right) \\
&\quad + \log \det 2\pi e (I_{N_1}) \\
&\quad + \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \begin{bmatrix} \sqrt{\rho_{11} \rho_{12}} H_{11} H_{12}^\dagger & \sqrt{\rho_{11}} H_{11} Q_{12} \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger & \sqrt{\rho_{12}} H_{12} Q_{12} \\ \sqrt{\rho_{12}} Q_{12}^\dagger H_{12}^\dagger & I_{M_2} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{11} \rho_{12}} H_{12} H_{11}^\dagger \\ \sqrt{\rho_{11}} Q_{12}^\dagger H_{11}^\dagger \end{bmatrix} \right) \\
&\quad + \log \det 2\pi e (I_{N_1}) - h(Z_1) - h(Z_2) \\
&\stackrel{(b)}{=} \log \det \left(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger + \rho_{12} H_{12} H_{12}^\dagger + \sqrt{\rho_{22} \rho_{12}} H_{22} Q_{12} H_{12}^\dagger + \sqrt{\rho_{22} \rho_{12}} H_{12} Q_{12}^\dagger H_{22}^\dagger \right) \\
&\quad + \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \begin{bmatrix} \sqrt{\rho_{11} \rho_{12}} H_{11} H_{12}^\dagger & \sqrt{\rho_{11}} H_{11} Q_{12} \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger & \sqrt{\rho_{12}} H_{12} Q_{12} \\ \sqrt{\rho_{12}} Q_{12}^\dagger H_{12}^\dagger & I_{M_2} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{11} \rho_{12}} H_{12} H_{11}^\dagger \\ \sqrt{\rho_{11}} Q_{12}^\dagger H_{11}^\dagger \end{bmatrix} \right), \tag{79}
\end{aligned}$$

where (a) is obtained by using Lemma 7 and Lemma 5 and (b) follows from the fact that $h(Z_i) = \log \det 2\pi e (I_{N_i})$, for $i = 1, 2$.

Sixth term: The sixth term is similar to the fifth term by replacing 1 and 2 in the indices.

APPENDIX B

PROOF OF ACHIEVABILITY FOR THEOREM 2

In this section, we prove the achievability for Theorem 2. More precisely, we will show the following.

Lemma 8. *For a given ρ and H , the feedback capacity region of a two-user MIMO Gaussian IC can*

achieve all rate pairs $(R_1, R_2) \in R_a(\overline{H}, \overline{\rho})$ such that

$$R_1 \leq \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger) - N_1, \quad (80)$$

$$R_2 \leq \log \det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger) - N_2, \quad (81)$$

$$R_1 \leq \log \det(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger) + \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^\dagger(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}\sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^\dagger) - N_1 - N_2, \quad (82)$$

$$R_2 \leq \log \det(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger) + \log \det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger - \sqrt{\rho_{22}\rho_{21}}H_{22}H_{21}^\dagger(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger)^{-1}\sqrt{\rho_{22}\rho_{21}}H_{21}H_{22}^\dagger) - N_1 - N_2, \quad (83)$$

$$R_1 + R_2 \leq \log \det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger) + \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^\dagger(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}\sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^\dagger) - N_1 - N_2, \quad (84)$$

$$R_1 + R_2 \leq \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger) + \log \det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger - \sqrt{\rho_{22}\rho_{21}}H_{22}H_{21}^\dagger(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger)^{-1}\sqrt{\rho_{22}\rho_{21}}H_{21}H_{22}^\dagger) - N_1 - N_2. \quad (85)$$

The rest of the section proves this result.

Lemma 9. *The feedback capacity region of the two-user discrete memoryless IC includes the set of (R_1, R_2) such that*

$$R_1 \leq I(U_2, X_1; Y_1), \quad (86)$$

$$R_2 \leq I(U_1, X_2; Y_2), \quad (87)$$

$$R_1 \leq I(U_1; Y_2|X_2) + I(X_1; Y_1 | U_1, U_2), \quad (88)$$

$$R_2 \leq I(U_2; Y_1|X_1) + I(X_2; Y_2 | U_1, U_2), \quad (89)$$

$$R_1 + R_2 \leq I(X_1; Y_1 | U_1, U_2) + I(U_1, X_2; Y_2), \quad (90)$$

$$R_1 + R_2 \leq I(X_2; Y_2 | U_1, U_2) + I(U_2, X_1; Y_1), \quad (91)$$

over all joint distributions $p(u_1)p(u_2)p(x_1|u_1)p(x_2|u_2)$.

Proof: This result follows from Lemma 1 of [8] with $U = 0$. ■

To achieve this rate region, the authors of [8] developed an infinite-staged achievable scheme that employs block Markov encoding, backward decoding, and Han-Kobayashi message splitting.

The rest of the section inner bounds this region to get the inner bound in Theorem 2. For this, we will

introduce some useful Lemmas.

Lemma 10. *The following holds for any $M_i \times N_j$ matrix S*

$$S(I_{N_j} + S^\dagger S)^{-1} S^\dagger \succeq 0. \quad (92)$$

Proof: It holds since it can be written as AEA^\dagger for $A = S$ and $E = (I_{N_j} + S^\dagger S)^{-1}$, which is p.s.d. because E is p.s.d.. ■

Lemma 11. *The following holds for any $M_i \times N_j$ matrix S*

$$\det(I_{N_j} + S^\dagger S - S^\dagger S(I_{N_j} + S^\dagger S)^{-1} S^\dagger S) \leq \det(2I_{N_j}). \quad (93)$$

Proof: Let us define $V \triangleq S^\dagger S$, we get

$$\begin{aligned} & \det(I_{N_j} + S^\dagger S - S^\dagger S(I_{N_j} + S^\dagger S)^{-1} S^\dagger S) \\ &= \det(I_{N_j} + S^\dagger (I_{M_i} - S E S^\dagger) S) \\ &= \det(I_{N_j} + S^\dagger (I_{M_i} - S(I_{N_j} + S^\dagger S)^{-1} S^\dagger) S) \\ &= \det(I_{N_j} + S^\dagger S - S^\dagger S(I_{N_j} + S^\dagger S)^{-1} S^\dagger S) \\ &= \det(I_{N_j} + V - V(I_{N_j} + V)^{-1} V) \\ &= \det(I_{N_j} + V - V(I_{N_j} + V)^{-1} (V + I_{N_j} - I_{N_j})) \\ &= \det(I_{N_j} + V - V(I_{N_j} - (I_{N_j} + V)^{-1})) \\ &= \det(I_{N_j} + V((I_{N_j} + V)^{-1})) \\ &= \det(I_{N_j} + (-I_{N_j} + I_{N_j} + V)((I_{N_j} + V)^{-1})) \\ &= \det(I_{N_j} + I_{N_j} - (I_{N_j} + V)^{-1}) \\ &\stackrel{(a)}{\leq} \det(2I_{N_j}), \end{aligned} \quad (94)$$

where (a) follows from the the fact that $V = S^\dagger S$ is p.s.d., and its eigenvalues are non-negative. So, the eigenvalues of $I_{N_j} + V$ are greater than or equal to 1. As a result, eigenvalues of $(I_{N_j} + V)^{-1}$, $(\lambda_1, \dots, \lambda_{N_j})$,

satisfy $0 \leq \lambda_k \leq 1$. So

$$\begin{aligned}
 \det(I_{N_j} + I_{N_j} - (I_{N_j} + V)^{-1}) &= (2 - \lambda_1), \dots, (2 - \lambda_{N_j}) \\
 &\leq 2^{N_j} \\
 &= \det(2I_{N_j}).
 \end{aligned} \tag{95}$$

This proves (93). ■

Now we present our achievability scheme. We use a power allocation according to

$$X_i = X_{ip} + X_{iu} \tag{96}$$

which states that signal is divided in two parts - private, and public which are independent of each other with $X_{ip} \sim \text{CN}(0, K_{X_{ip}})$ being the private part and $X_{iu} \sim \text{CN}(0, K_{X_{iu}})$ being public part such that:

$$K_{X_{ip}} = I_{M_i} - \sqrt{\rho_{ij}} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} \sqrt{\rho_{ij}} H_{ij}, \tag{97}$$

and

$$K_{X_{iu}} = I_{M_i} - K_{X_{ip}}. \tag{98}$$

We note that the power allocation is feasible since $I_{M_i} - K_{X_{ip}} \succeq 0$ by Lemma 10 substituting $\sqrt{\rho_{ij}} H_{ij}^\dagger$ into S .

We will now expand the achievability in Lemma 9 using $U_i = X_{iu}$ for $i \in \{1, 2\}$. Before expanding each term in Lemma 9, we evaluate some entropies as follows.

$$h(Y_i) = \log \det(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} H_{ji}^\dagger) \quad , \tag{99}$$

and

$$h(Y_i | X_i) = \log \det \left(I_{N_i} + \rho_{ji} H_{ji} H_{ji}^\dagger \right) \quad . \tag{100}$$

In addition, we have

$$\begin{aligned}
& h(Y_i | U_i, U_j) \\
& \geq h(Y_i | U_i, U_j, X_j) \\
& = \log \det(I_{N_i} + \rho_{ii} H_{ii} K_{X_{ip}} H_{ii}^\dagger) \\
& = \log \det(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger - \sqrt{\rho_{ii} \rho_{ij}} H_{ii} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} \sqrt{\rho_{ii} \rho_{ij}} H_{ij} H_{ii}^\dagger). \tag{101}
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
h(Y_i | U_j, X_i) & \leq \log \det(I_{N_i} + \rho_{ji} H_{ji} K_{X_{jp}} H_{ji}^\dagger) \\
& \stackrel{(a)}{\leq} \log \det(2I_{N_i}) \\
& = N_i, \tag{102}
\end{aligned}$$

where (a) follows from Lemma 11 by substituting $\sqrt{\rho_{ji}} H_{ji}^\dagger$ in S . This shows that $h(Y_i | U_j, X_i)$ is upper-bounded by N_i .

In our achievability, $h(Y_i | U_j, X_i)$ appeared with a minus sign. So, without loss of generality we can replace it with its bound N_i for the achievability.

The rest of the section considers the six terms in Lemma 9 and uses each of them to get the terms in the inner-bound of Lemma 8.

First term: For the first term in Lemma 9, we have

$$\begin{aligned}
& I(U_2, X_1; Y_1) \\
& = h(Y_1) - h(Y_1 | U_2, X_1) \\
& \stackrel{(a)}{=} \log \det(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger + \rho_{21} H_{21} H_{21}^\dagger) - h(Y_1 | U_2, X_1) \\
& \stackrel{(b)}{\geq} \log \det(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger + \rho_{21} H_{21} H_{21}^\dagger) - N_1, \tag{103}
\end{aligned}$$

where (a) follows from (99) and (b) follows from (102).

Second term: The second bound is similar to the first bound by replacing 1 and 2 in the indices.

Third term: For the third bound in Lemma 9, we have

$$\begin{aligned}
& I(U_1; Y_2 | X_2) + I(X_1; Y_1 | U_1, U_2) \\
&= h(Y_2 | X_2) - h(Y_2 | U_1, X_2) + h(Y_1 | U_1, U_2) - h(Y_1 | U_1, U_2, X_1) \\
&\geq h(Y_2 | X_2) - h(Y_2 | U_1, X_2) + h(Y_1 | U_1, U_2, X_2) - h(Y_1 | U_1, U_2, X_1) \\
&\stackrel{(a)}{=} \log \det \left(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger \right) + \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \sqrt{\rho_{11} \rho_{12}} H_{11} H_{12}^\dagger \right. \\
&\quad \left. (I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger)^{-1} \sqrt{\rho_{11} \rho_{12}} H_{12} H_{11}^\dagger \right) - h(Y_2 | U_1, X_2) - h(Y_1 | U_1, U_2, X_1) \\
&\stackrel{(b)}{\geq} \log \det \left(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger \right) + \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \sqrt{\rho_{11} \rho_{12}} H_{11} H_{12}^\dagger \right. \\
&\quad \left. (I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger)^{-1} \sqrt{\rho_{11} \rho_{12}} H_{12} H_{11}^\dagger \right) - N_1 - N_2, \tag{104}
\end{aligned}$$

where (a) is obtained from (100) and (101) and (b) follows from (102).

Fourth term: The fourth term is similar to the third term by replacing 1 and 2 in the indices.

Fifth term: For the fifth bound in Lemma 9, we have

$$\begin{aligned}
& I(X_1; Y_1 | U_1, U_2) + I(U_1, X_2; Y_2) \\
&= h(Y_1 | U_1, U_2) - h(Y_1 | U_1, U_2, X_1) + h(Y_2) - h(Y_2 | U_1, X_2) \tag{105}
\end{aligned}$$

$$\geq h(Y_1 | U_1, U_2, X_2) - h(Y_1 | U_1, U_2, X_1) + h(Y_2) - h(Y_2 | U_1, X_2) \tag{106}$$

$$\begin{aligned}
&\stackrel{(a)}{=} \log \det(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger + \rho_{12} H_{12} H_{12}^\dagger) + \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \sqrt{\rho_{11} \rho_{12}} H_{11} H_{12}^\dagger \right. \\
&\quad \left. (I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger)^{-1} \sqrt{\rho_{11} \rho_{12}} H_{12} H_{11}^\dagger \right) - h(Y_2 | U_1, X_2) - h(Y_1 | U_1, U_2, X_1) \tag{107}
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(b)}{\geq} \log \det(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger + \rho_{12} H_{12} H_{12}^\dagger) + \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \sqrt{\rho_{11} \rho_{12}} H_{11} H_{12}^\dagger \right. \\
&\quad \left. (I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger)^{-1} \sqrt{\rho_{11} \rho_{12}} H_{12} H_{11}^\dagger \right) - N_1 - N_2, \tag{108}
\end{aligned}$$

where (a) is obtained from (99) and (101), and (b) follows from (102).

Sixth term: The sixth term is similar to the fifth term by replacing 1 and 2 in the indices.

APPENDIX C

PROOF OF OUTER BOUND FOR THEOREM 2

In this section, we prove cross covariance matrix $Q = 0$ is approximately optimal for the capacity region of MIMO IC with feedback. As mentioned in Section III, it is enough to prove that

$$\mathcal{R}_o(Q) \subseteq \mathcal{R}_o(0) \oplus ([0, N_1] \times [0, N_2]), \quad (109)$$

for any cross-covariance matrix Q .

Now, we give three important relations that would be used in the main proof.

Define $E \triangleq (I_{N_2} + \sqrt{\rho_{ij}}H_{ij}(I - Q_{ij}Q_{ij}^\dagger)\sqrt{\rho_{ij}}H_{ij}^\dagger)^{-1}$. The first relation is as follows

$$\begin{aligned} & I_{N_i} + \rho_{ii}H_{ii}H_{ii}^\dagger - \\ & \left[\begin{array}{cc} \sqrt{\rho_{ii}\rho_{ij}}H_{ii}H_{ij}^\dagger & \sqrt{\rho_{ii}}H_{ii}Q_{ij} \end{array} \right] \left[\begin{array}{cc} I_{N_j} + \rho_{ij}H_{ij}H_{ij}^\dagger & \sqrt{\rho_{ij}}H_{ij}Q_{ij} \\ \sqrt{\rho_{ij}}Q_{ij}^\dagger H_{ij}^\dagger & I_{M_j} \end{array} \right]^{-1} \left[\begin{array}{c} \sqrt{\rho_{ii}\rho_{ij}}H_{ij}H_{ii}^\dagger \\ \sqrt{\rho_{ii}}Q_{ij}^\dagger H_{ii}^\dagger \end{array} \right] \\ = & I_{N_i} + \rho_{ii}H_{ii}(I_{M_i} - \\ & \left[\begin{array}{cc} \sqrt{\rho_{ij}}H_{ij}^\dagger & Q_{ij} \end{array} \right] \left[\begin{array}{cc} I_{N_j} + \rho_{ij}H_{ij}H_{ij}^\dagger & \sqrt{\rho_{ij}}H_{ij}Q_{ij} \\ \sqrt{\rho_{ij}}Q_{ij}^\dagger H_{ij}^\dagger & I_{M_j} \end{array} \right]^{-1} \left[\begin{array}{c} \sqrt{\rho_{ij}}H_{ij} \\ Q_{ij}^\dagger \end{array} \right])H_{ii}^\dagger \\ \stackrel{(a)}{=} & I_{N_i} + \rho_{ii}H_{ii}(I_{M_i} - \\ & \left[\begin{array}{cc} \sqrt{\rho_{ij}}H_{ij}^\dagger & Q_{ij} \end{array} \right] \left[\begin{array}{cc} E & -E\sqrt{\rho_{ij}}H_{ij}Q_{ij} \\ -\sqrt{\rho_{ij}}Q_{ij}^\dagger H_{ij}^\dagger E & I_{M_j} + \sqrt{\rho_{ij}}Q_{ij}^\dagger H_{ij}^\dagger E\sqrt{\rho_{ij}}H_{ij}Q_{ij} \end{array} \right] \\ & \left[\begin{array}{c} \sqrt{\rho_{ij}}H_{ij} \\ Q_{ij}^\dagger \end{array} \right])H_{ii}^\dagger \\ \stackrel{(b)}{=} & I_{N_i} + \rho_{ii}H_{ii}(I - Q_{ij}Q_{ij}^\dagger - (I - Q_{ij}Q_{ij}^\dagger)\sqrt{\rho_{ij}}H_{ij}^\dagger E\sqrt{\rho_{ij}}H_{ij}(I - Q_{ij}Q_{ij}^\dagger))H_{ii}^\dagger \\ \stackrel{(c)}{=} & I_{N_i} + \rho_{ii}H_{ii}(L(I - Q_{ij}Q_{ij}^\dagger, \sqrt{\rho_{ij}}H_{ij}^\dagger))H_{ii}^\dagger \\ \stackrel{(d)}{\leq} & I_{N_i} + \rho_{ii}H_{ii}(L(I, \sqrt{\rho_{ij}}H_{ij}^\dagger))H_{ii}^\dagger, \end{aligned} \quad (110)$$

where $L(K, S)$ is as in (54), (a) follows since the inverse can be verified easily, (b) follows from finding the product of matrices, (c) follows from the definition of $L(K, S)$ in (54), and (d) follows from Lemma 4.

The second relation is as follows

$$\begin{aligned}
& \log \det \left(I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger - \rho_{ij} H_{ij} Q_{ij} Q_{ij}^\dagger H_{ij}^\dagger \right) \\
& \leq \log \det \left(I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger \right) .
\end{aligned} \tag{111}$$

The third relation is as follows

$$\begin{aligned}
& \log \det (I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ii} Q_{ij} H_{ji}^\dagger + \sqrt{\rho_{ii} \rho_{ji}} H_{ji} Q_{ij}^\dagger H_{ii}^\dagger) \\
& \stackrel{(a)}{\leq} \log \det (I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} H_{ji}^\dagger + \rho_{ii} H_{ii} Q_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} Q_{jj} H_{ji}^\dagger) \\
& \stackrel{(b)}{\leq} \log \det (I_{N_i} + 2\rho_{ii} H_{ii} H_{ii}^\dagger + 2\rho_{ji} H_{ji} H_{ji}^\dagger) \\
& \leq \log \det (I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} H_{ji}^\dagger) + N_i,
\end{aligned} \tag{112}$$

where (a) follows from $(A - B)(A^\dagger - B^\dagger) = AA^\dagger + BB^\dagger - AB^\dagger - BA^\dagger \succeq 0$ by substituting $\sqrt{\rho_{ii}} H_{ii} X_i$ and $\sqrt{\rho_{ji}} H_{ji} X_j$ in A and B , respectively, (b) follows from the fact that $I \succeq Q_{ii}$.

Thus, we proved that among these three expansions, the first two expansions we started with are maximized by $Q_{ij} = 0$ while the third one is outer-bounded by the corresponding expression with $Q_{ij} = 0$ plus N_1 .

Now, we consider each of the six expressions in the definition of the region $\mathcal{R}_o(Q)$ and outer-bound each expression to find the gap with $\mathcal{R}_o(0)$ being constant thus proving that $\mathcal{R}_o(Q) \subseteq \mathcal{R}_o(0) \oplus ([0, N_1] \times [0, N_2])$ which proves the result.

Let the right-hand sides of the six expressions in the definition of $\mathcal{R}_0(Q)$ in (6)-(11) be labeled as $I_1(Q)$, $I_2(Q)$, $I_3(Q)$, $I_4(Q)$, $I_5(Q)$, and $I_6(Q)$ respectively. Then, the constant gap outer-bound is shown in the following Lemma.

Lemma 12. *We have*

$$I_1(Q) \leq I_1(0) + N_1, \tag{113}$$

$$I_2(Q) \leq I_2(0) + N_2, \tag{114}$$

$$I_3(Q) \leq I_3(0), \tag{115}$$

$$I_4(Q) \leq I_4(0), \tag{116}$$

$$I_5(Q) \leq I_5(0) + N_2, \tag{117}$$

$$I_6(Q) \leq I_6(0) + N_1. \tag{118}$$

Proof: We start with (113).

$$\begin{aligned}
I_1(Q) &= \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger + \sqrt{\rho_{11}\rho_{21}}H_{11}QH_{21}^\dagger + \sqrt{\rho_{11}\rho_{21}}H_{21}Q^\dagger H_{11}^\dagger) \\
&\stackrel{(a)}{\leq} \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger) + N_1 \\
&= I_1(0) + N_1,
\end{aligned} \tag{119}$$

where (a) follows from (112).

Proof of (114) is similar to (113) by replacing 1 and 2 in the indices.

For the proof of (115) we have,

$$\begin{aligned}
I_3(Q) &= \log \det \left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger - \rho_{12}H_{12}QQ^\dagger H_{12}^\dagger \right) + \log \det \left(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \right. \\
&\quad \left[\begin{array}{cc} \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^\dagger & \sqrt{\rho_{11}}H_{11}Q \end{array} \right] \left[\begin{array}{cc} I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger & \sqrt{\rho_{12}}H_{12}Q \\ \sqrt{\rho_{12}}Q^\dagger H_{12}^\dagger & I_{M_2} \end{array} \right]^{-1} \\
&\quad \left. \left[\begin{array}{c} \sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^\dagger \\ \sqrt{\rho_{11}}Q^\dagger H_{11}^\dagger \end{array} \right] \right) \\
&\stackrel{(a)}{\leq} \log \det \left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger \right) + \log \det \left(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \right. \\
&\quad \left. \rho_{11}\rho_{12}H_{11}H_{12}^\dagger (I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1} H_{12}H_{11}^\dagger \right) \\
&= I_3(0),
\end{aligned} \tag{120}$$

where (a) follows since the first expression is outer-bounded as in (111) and the outer-bound for the second expression can be shown on similar lines as (110).

Proof of (116) is similar to (115) by replacing 1 and 2 in the indices.

For the proof of (117) we have

$$\begin{aligned}
I_5(Q) &= \log \det \left(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger + \rho_{12} H_{12} H_{12}^\dagger + \sqrt{\rho_{22} \rho_{12}} H_{22} Q^\dagger H_{12}^\dagger + \sqrt{\rho_{22} \rho_{12}} H_{12} Q H_{22}^\dagger \right) \\
&\quad + \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \begin{bmatrix} \sqrt{\rho_{11} \rho_{12}} H_{11} H_{12}^\dagger & \sqrt{\rho_{11}} H_{11} Q \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger & \sqrt{\rho_{12}} H_{12} Q \\ \sqrt{\rho_{12}} Q^\dagger H_{12}^\dagger & I_{M_2} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\rho_{11} \rho_{12}} H_{12} H_{11}^\dagger \\ \sqrt{\rho_{11}} Q^\dagger H_{11}^\dagger \end{bmatrix} \right) \\
&\stackrel{(a)}{\leq} \log \det(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger + \rho_{12} H_{12} H_{12}^\dagger) + \log \det \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \right. \\
&\quad \left. \rho_{11} \rho_{12} H_{11} H_{12}^\dagger (I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger)^{-1} H_{12} H_{11}^\dagger \right) + N_2 \\
&= I_5(0) + N_2,
\end{aligned} \tag{121}$$

where (a) follows from (112) and using similar steps as in (110).

Proof of (118) is similar to (117) by replacing 1 and 2 in the indices. ■

APPENDIX D

PROOF OF RECIPROCITY IN $\mathcal{R}_o(0)$

In this section, we prove that replacing \overline{H} and $\overline{\rho}$ by \overline{H}^R and $\overline{\rho}^R$, respectively, and interchanging M and N for antennas at the nodes gives the same expressions in $\mathcal{R}_o(0)$.

We shall prove this in two steps. In the first step we shall prove

$$\mathcal{R}_o(\overline{H}, \overline{\rho}) = \mathcal{R}_o(\overline{H}', \overline{\rho}^R), \tag{122}$$

where $\overline{H}' = \{H_{11}^\dagger, H_{21}^\dagger, H_{12}^\dagger, H_{22}^\dagger\}$ and in the second step we shall prove that

$$\mathcal{R}_o(\overline{H}', \overline{\rho}^R) = \mathcal{R}_o(\overline{H}^R, \overline{\rho}^R). \tag{123}$$

Clearly, the above two equalities prove the lemma.

Let the right-hand sides of the six expressions in the definition of $\mathcal{R}_o(0)$ in (13)-(18) be labeled as I_1 , I_2 , I_3 , I_4 , I_5 , and I_6 respectively.

First Step: In this step, we prove that:

$$I_1 = I'_3, \quad (124)$$

$$I_2 = I'_4, \quad (125)$$

$$I_3 = I'_1, \quad (126)$$

$$I_4 = I'_2, \quad (127)$$

$$I_5 = I'_6, \quad (128)$$

$$I_6 = I'_5, \quad (129)$$

where I'_k is obtained from I_k by interchanging M and N , replacing H_{ij} with H_{ji}^\dagger , and replacing ρ_{ij} with ρ_{ji} .

Since I_1 and I_3 are both bounds for R_1 , I_2 and I_4 are both bounds for R_2 , and I_5 and I_6 are both bounds for $R_1 + R_2$, (124)-(129) will prove that $\mathcal{R}_o(\overline{H}, \overline{\rho}) = \mathcal{R}_o(\overline{H}', \overline{\rho}^R)$.

We start with proving (124). For simplicity we define $K \triangleq (I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger)^{-1}$, $K' \triangleq (I_{M_1} +$

$\rho_{21}H_{21}^\dagger H_{21})^{-1}$, and $L \triangleq \rho_{11}H_{11}H_{11}^\dagger$. We get

$$\begin{aligned}
I_1 &= \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger) \\
&= \log \det(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger) + \log \det(I_{N_1} + K\rho_{11}H_{11}H_{11}^\dagger) \\
&= \log \det(K^{-1}) + \log \det(I_{N_1} + KL) \\
&\stackrel{(a)}{=} \log \det(K^{-1}) + \log \det(I_{N_1} + LK) \\
&= \log \det(K^{-1}) + \log \det(I_{N_1} + LKI) \\
&= \log \det(K^{-1}) + \log \det(I_{N_1} + LK(I + \rho_{21}H_{21}H_{21}^\dagger - \rho_{21}H_{21}H_{21}^\dagger)) \\
&= \log \det(K^{-1}) + \log \det(I_{N_1} + LK(I + \rho_{21}H_{21}H_{21}^\dagger - \rho_{21}H_{21}(I)H_{21}^\dagger)) \\
&= \log \det(K^{-1}) + \log \det(I_{N_1} + LK(I + \rho_{21}H_{21}H_{21}^\dagger - \rho_{21}H_{21}(K'^{-1}K')H_{21}^\dagger)) \\
&= \log \det(K^{-1}) + \\
&\quad \log \det(I_{N_1} + LK(I + \rho_{21}H_{21}H_{21}^\dagger - \rho_{21}H_{21}((I + \rho_{21}H_{21}H_{21}^\dagger)K')H_{21}^\dagger)) \\
&= \log \det(K^{-1}) + \\
&\quad \log \det(I_{N_1} + LK((I + \rho_{21}H_{21}H_{21}^\dagger) - \rho_{21}((I + \rho_{21}H_{21}H_{21}^\dagger)H_{21}K')H_{21}^\dagger)) \\
&= \log \det(K^{-1}) + \log \det(I_{N_1} + LK(K^{-1} - \rho_{21}K^{-1}H_{21}K')H_{21}^\dagger) \\
&= \log \det(I + \rho_{21}H_{21}^\dagger H_{21}) + \log \det(I_{N_1} + L(I - \rho_{21}H_{21}K')H_{21}^\dagger) \\
&\stackrel{(b)}{=} \log \det(I + \rho_{21}H_{21}^\dagger H_{21}) + \log \det(I + L - L\rho_{21}H_{21}K'H_{21}^\dagger) \\
&= \log \det(I + \rho_{21}H_{21}^\dagger H_{21}) + \\
&\quad \log \det(I + \rho_{11}H_{11}H_{11}^\dagger - \rho_{11}\rho_{21}H_{11}H_{11}^\dagger H_{21}(I + \rho_{21}H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger) \\
&\stackrel{(c)}{=} \log \det(I + \rho_{21}H_{21}^\dagger H_{21}) + \\
&\quad \log \det(I + \rho_{11}H_{11}^\dagger H_{11} - \rho_{11}\rho_{21}H_{11}^\dagger H_{21}(I + \rho_{21}H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger H_{11}) \tag{131}
\end{aligned}$$

$$= I'_3, \tag{132}$$

where (a), (b) and (c) follow from Sylvester's determinant theorem [20]. (125) can be proved similarly due to symmetry. In addition, (126) and (127) can be obtained in the reverse direction similarly.

We move toward the proof of (128). We should prove

$$I_5 = I'_6, \tag{133}$$

where

$$\begin{aligned} I_5 = & \log \det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger) + \\ & \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \rho_{11}\rho_{12}H_{11}H_{12}^\dagger(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}H_{12}H_{11}^\dagger), \end{aligned} \quad (134)$$

and

$$\begin{aligned} I'_6 = & \log \det(I_{M_1} + \rho_{11}H_{11}^\dagger H_{11} + \rho_{12}H_{12}^\dagger H_{12}) + \\ & \log \det(I_{M_2} + \rho_{22}H_{22}^\dagger H_{22} - \rho_{22}\rho_{12}H_{22}^\dagger H_{12}(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12})^{-1}H_{12}^\dagger H_{22}). \end{aligned} \quad (135)$$

If we define

$$a \triangleq \log \det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger), \quad (136)$$

$$b \triangleq \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \rho_{11}\rho_{12}H_{11}H_{12}^\dagger(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}H_{12}H_{11}^\dagger), \quad (137)$$

$$c \triangleq \log \det(I_{M_1} + \rho_{11}H_{11}^\dagger H_{11} + \rho_{12}H_{12}^\dagger H_{12}), \quad (138)$$

$$d \triangleq \log \det(I_{M_2} + \rho_{22}H_{22}^\dagger H_{22} - \rho_{22}\rho_{12}H_{22}^\dagger H_{12}(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12})^{-1}H_{12}^\dagger H_{22}), \quad (139)$$

then, it is sufficient to prove $a + b = c + d$ or $a - d = c - b$.

Since (130) is equal to (131), we have

$$\begin{aligned} & \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger) - \\ & \log \det(I + \rho_{11}H_{11}^\dagger H_{11} - \rho_{11}\rho_{21}H_{11}^\dagger H_{21}(I + \rho_{21}H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger H_{11}) = \\ & \log \det(I + \rho_{21}H_{21}^\dagger H_{21}). \end{aligned} \quad (140)$$

Using similar method, we can see that

$$a - d = \log \det(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12}), \quad (141)$$

and

$$c - b = \log \det(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger), \quad (142)$$

which according to Sylvester's determinant theorem [20] are equal. This proves the $I_5 = I'_6$.

(129) can be proved similar to the proof of (128) due to symmetry.

Second Step: It can be proved with a similar discussion as in Appendix E of [4]. A brief sketch of the proof is given below for completeness.

Suppose S is a p.s.d. matrix and S^* represents its complex conjugate, i.e., the matrix obtained by replacing all its entries by the corresponding complex conjugates. Then, it is easy to see that

$$\log \det(I + S) = \log \det(I + S^*). \quad (143)$$

However, note that all the terms in the different bounds of $\mathcal{R}_o(0)$ are of the form of $\log \det(I + S)$. This in turn proves that if we replace all the channel matrices of a two-user MIMO IC with feedback by their complex conjugates the set of upper bounds remain the same. From this fact, it easily follows that

$$\mathcal{R}_o(\overline{H}', \overline{\rho}^R) = \mathcal{R}_o(\overline{H}^R, \overline{\rho}^R). \quad (144)$$

APPENDIX E

PROOF OF THEOREM 5

In this section, we will find the limit of $\mathcal{R}_o(0)/\log \text{SNR}$ as $\text{SNR} \rightarrow \infty$ to get the result as in the statement of the Theorem 5. This follows from Theorem 3 since the capacity region is inner and outer-bounded by $\mathcal{R}_o(0)$ with constant gaps which would vanish for the degrees of freedom.

Before going over each of the terms in $\mathcal{R}_o(0)$ and finding its high SNR limit, we first give some Lemmas that will be used for the proof of the Theorem.

Lemma 13 ([4]). *Let $H_{ij} \in \mathbb{C}^{N_j \times M_i}$ be a full rank channel matrix. Then, the following holds*

$$\log \det \left(I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger \right) = \alpha_{ij} \min(M_i, N_j) \log \text{SNR} + o(\log \text{SNR}). \quad (145)$$

Lemma 14. *Let $\Sigma \in \mathbb{C}^{N \times M}$ be a diagonal matrix with elements $\sigma_1, \dots, \sigma_m$ where $m = \min(M, N)$ and $\Lambda \in \mathbb{C}^{m \times m}$ be a diagonal matrix with elements $|\sigma_1|^2, \dots, |\sigma_m|^2$, then*

$$\begin{aligned} & \Sigma^\dagger \begin{bmatrix} (I_m + \Lambda)^{-1} & 0 \\ 0 & I_{(N-M)^+} \end{bmatrix} \Sigma \\ &= \begin{bmatrix} I_m - (I_m + \Lambda)^{-1} & 0 \\ 0 & 0_{(M-N)^+} \end{bmatrix}. \end{aligned} \quad (146)$$

Proof: We will split the proof in two cases, depending on whether $M \geq N$ or $M < N$.

Case 1 - $M \geq N$: In this case, we have

$$\begin{aligned}
& \Sigma^\dagger \begin{bmatrix} (I_m + \Lambda)^{-1} & 0 \\ 0 & I_{(N-M)^+} \end{bmatrix} \Sigma \\
&= \begin{bmatrix} \sigma_1^* & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_m^* \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{1+|\sigma_1|^2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{1+|\sigma_m|^2} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \sigma_m & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{|\sigma_1|^2}{1+|\sigma_1|^2} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \frac{|\sigma_m|^2}{1+|\sigma_m|^2} & 0 \\ 0 & 0 & 0 & 0_{(M-N)^+} \end{bmatrix} \\
&= \begin{bmatrix} 1 - \frac{1}{1+|\sigma_1|^2} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 - \frac{1}{1+|\sigma_m|^2} & 0 \\ 0 & 0 & 0 & 0_{(M-N)^+} \end{bmatrix} \\
&= \begin{bmatrix} I_m - (I_m + \Lambda)^{-1} & 0 \\ 0 & 0_{(M-N)^+} \end{bmatrix}. \tag{147}
\end{aligned}$$

Case 2 - $M < N$: In this case, we have

$$\begin{aligned}
& \Sigma^\dagger \begin{bmatrix} (I_m + \Lambda)^{-1} & 0 \\ 0 & I_{(N-M)^+} \end{bmatrix} \Sigma \\
&= \begin{bmatrix} \sigma_1^* & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \sigma_m^* & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{1+|\sigma_1|^2} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \frac{1}{1+|\sigma_m|^2} & 0 \\ 0 & 0 & 0 & I_{(N-M)^+} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_m \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{|\sigma_1|^2}{1+|\sigma_1|^2} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \frac{|\sigma_m|^2}{1+|\sigma_m|^2} & 0 \\ 0 & 0 & 0 & 0_{(M-N)^+} \end{bmatrix} \\
&= \begin{bmatrix} 1 - \frac{1}{1+|\sigma_1|^2} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 - \frac{1}{1+|\sigma_m|^2} & 0 \\ 0 & 0 & 0 & 0_{(M-N)^+} \end{bmatrix} \\
&= \begin{bmatrix} I_m - (I_m + \Lambda)^{-1} & 0 \\ 0 & 0_{(M-N)^+} \end{bmatrix}. \tag{148}
\end{aligned}$$

■

Lemma 15 ([4]). Let $H_{ii} \in \mathbb{C}^{N_i \times M_i}$ and $H_{ji} \in \mathbb{C}^{N_i \times M_j}$ be two full rank channel matrices such that $[H_{ii} H_{ji}]$ is also full rank. Then, the following holds

$$\log \det(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} H_{ji}^\dagger) = f(N_i, (\alpha_{ii}, M_i), (\alpha_{ji}, M_j)) \log \text{SNR} + o(\log \text{SNR}) \tag{149}$$

where f is defined in 28.

Lemma 16. Let $H_{ii} \in \mathbb{C}^{N_i \times M_i}$ and $H_{ij} \in \mathbb{C}^{N_i \times M_j}$ be two channel matrices with each entry independently chosen from $\text{CN}(0, 1)$. Then, the following holds with probability 1 (over the randomness of channel

matrices).

$$\begin{aligned}
& \log \det (I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger - \sqrt{\rho_{ii} \rho_{ij}} H_{ii} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} \sqrt{\rho_{ii} \rho_{ij}} H_{ij} H_{ii}^\dagger) \\
&= [\alpha_{ii} \min((M_i - N_j)^+, N_i) + (\alpha_{ii} - \alpha_{ij})^+ (\min(M_i, N_i) - \min((M_i - N_j)^+, N_i))] \log \text{SNR} \\
&+ o(\log \text{SNR}).
\end{aligned} \tag{150}$$

Proof: Let the singular value decomposition (SVD) of the channel matrix H_{ij} be given by $H_{ij} = V_{ij} \Sigma_{ij} U_{ij}^\dagger$, where $V_{ij} \in U^{N_j \times N_j}$ and $U_{ij} \in U^{M_i \times M_i}$ are unitary matrices and $\Sigma_{ij} \in U^{N_j \times M_i}$ is a rectangular matrix containing the singular values along its diagonal. Using the SVD of the matrix H_{ij} we get

$$\begin{aligned}
& I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger - \sqrt{\rho_{ii} \rho_{ij}} H_{ii} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} \sqrt{\rho_{ii} \rho_{ij}} H_{ij} H_{ii}^\dagger \\
&= I_{N_i} + \rho_{ii} H_{ii} \left(I_{M_i} - \rho_{ij} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} H_{ij} \right) H_{ii}^\dagger
\end{aligned} \tag{151}$$

$$\begin{aligned}
& \stackrel{(a)}{=} I_{N_i} + \rho_{ii} H_{ii} (I_{M_i} - \rho_{ij} H_{ij}^\dagger V_{ij}) \begin{bmatrix} (I_{m_{ij}} + \text{SNR}^{\alpha_{ij}} \Lambda_{ij})^{-1} & 0 \\ 0 & I_{(N_j - M_i)^+} \end{bmatrix} V_{ij}^\dagger H_{ij} H_{ii}^\dagger \\
&= I_{N_i} + \rho_{ii} H_{ii} (I_{M_i} - \rho_{ij} U_{ij} \Sigma_{ij}^\dagger) \begin{bmatrix} (I_{m_{ij}} + \text{SNR}^{\alpha_{ij}} \Lambda_{ij})^{-1} & 0 \\ 0 & I_{(N_j - M_i)^+} \end{bmatrix} \Sigma_{ij} U_{ij}^\dagger H_{ii}^\dagger \\
&= I_{N_i} + \rho_{ii} H_{ii} U_{ij} (I_{M_i} - \text{SNR}^{\alpha_{ij}} \Sigma_{ij}^\dagger) \begin{bmatrix} (I_{m_{ij}} + \text{SNR}^{\alpha_{ij}} \Lambda_{ij})^{-1} & 0 \\ 0 & I_{(N_j - M_i)^+} \end{bmatrix} \Sigma_{ij} U_{ij}^\dagger H_{ii}^\dagger \\
& \stackrel{(b)}{=} I_{N_i} + \rho_{ii} H_{ii} U_{ij} (I_{M_i} - \begin{bmatrix} I_{m_{ij}} - (I_{m_{ij}} + \text{SNR}^{\alpha_{ij}} \Lambda_{ij})^{-1} & 0 \\ 0 & 0_{(M_i - N_j)^+} \end{bmatrix}) U_{ij}^\dagger H_{ii}^\dagger \\
&= I_{N_i} + \text{SNR}^{\alpha_{ii}} H_{ii} U_{ij} \begin{bmatrix} (I_{m_{ij}} + \text{SNR}^{\alpha_{ij}} \Lambda_{ij})^{-1} & 0 \\ 0 & I_{(M_i - N_j)^+} \end{bmatrix} U_{ij}^\dagger H_{ii}^\dagger,
\end{aligned} \tag{152}$$

where (a) results from SVD of the matrix H_{ij} and (b) follows from Lemma 14.

Let us decompose $U_{ij} \in U^{M_i \times M_i}$ into two parts, U_{ij1} and U_{ij2} such that $U_{ij} = [U_{ij1} \ U_{ij2}]$, where $U_{ij1} \in U^{M_i \times \min\{M_i, N_j\}}$ and $U_{ij2} \in U^{M_i \times (M_i - N_j)^+}$. Then, we get

$$\begin{aligned}
& \log \det(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger \sqrt{\rho_{ii} \rho_{ij}} H_{ii} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} \sqrt{\rho_{ii} \rho_{ij}} H_{ij} H_{ii}^\dagger) \\
&= \log \det(I_{N_i} + \text{SNR}^{\alpha_{ii}} H_{ii} (U_{ij} \begin{bmatrix} (I_{m_{ij}} + \text{SNR}^{\alpha_{ij}} \Lambda_{ij})^{-1} & 0 \\ 0 & I_{(M_i - N_j)^+} \end{bmatrix} U_{ij}^\dagger) H_{ii}^\dagger) \\
&= \log \det(I_{N_i} + H_{ii} (\text{SNR}^{\alpha_{ii}} U_{ij1} (I_{m_{ij}} + \text{SNR}^{\alpha_{ij}} \Lambda_{ij})^{-1} U_{ij1}^\dagger + \text{SNR}^{\alpha_{ii}} (U_{ij2} U_{ij2}^\dagger)) H_{ii}^\dagger) \\
&= \log \det(I_{N_i} + \text{SNR}^{\alpha_{ii}} H_{ii} U_{ij2} U_{ij2}^\dagger H_{ii}^\dagger + \text{SNR}^{\alpha_{ii} - \alpha_{ij}} H_{ii} U_{ij1} (\text{SNR}^{-\alpha_{ij}} I_{m_{ij}} + \Lambda_{ij})^{-1} U_{ij1}^\dagger H_{ii}^\dagger), \quad (153)
\end{aligned}$$

where $m_{ij} = \min(M_i, N_j)$, Λ_{ij} is a diagonal matrix containing the non-zero eigenvalues of $H_{ij} H_{ij}^\dagger$.

We note that Λ_{ij} is invertible and when SNR is large, we can bound $\text{SNR}^{-\alpha_{ij}} I_{m_{ij}} + \Lambda_{ij}$ from above and below as, $\Lambda_{ij} \preceq \text{SNR}^{-\alpha_{ij}} I_{m_{ij}} + \Lambda_{ij} \preceq I + \Lambda_{ij}$. We will only pursue the direction where $\text{SNR}^{-\alpha_{ij}} I_{m_{ij}} + \Lambda_{ij} \succeq \Lambda_{ij}$ and can see that both the directions produce the same result and thus replacing the inner and outer bound by equality. In what follows, even though $\text{SNR}^{-\alpha_{ij}} I_{m_{ij}} + \Lambda_{ij} \succeq \Lambda_{ij}$, we will substitute $\text{SNR}^{-\alpha_{ij}} I_{m_{ij}} + \Lambda_{ij} = \Lambda_{ij}$ since by the inner and outer-bounding approach, it can be seen the the limit will be exactly the same thus not causing any difference in the result. Thus, we have

$$\begin{aligned}
& \log \det(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger \sqrt{\rho_{ii} \rho_{ij}} H_{ii} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} \sqrt{\rho_{ii} \rho_{ij}} H_{ij} H_{ii}^\dagger) \\
&= \log \det(I_{N_i} + \text{SNR}^{\alpha_{ii}} H_{ii} U_{ij2} U_{ij2}^\dagger H_{ii}^\dagger + \text{SNR}^{(\alpha_{ii} - \alpha_{ij})} H_{ii} U_{ij1} (\Lambda_{ij})^{-1} U_{ij1}^\dagger H_{ii}^\dagger + o(\log \text{SNR})) \\
&\stackrel{(a)}{=} \log \det(I_{N_i} + \text{SNR}^{\alpha_{ii}} H_{ii} U_{ij2} U_{ij2}^\dagger H_{ii}^\dagger + \text{SNR}^{(\alpha_{ii} - \alpha_{ij})^+} H_{ii} U_{ij1} (\Lambda_{ij})^{-1} U_{ij1}^\dagger H_{ii}^\dagger) \\
&\stackrel{(b)}{=} f(N_i, (\alpha_{ii}, (M_i - N_j)^+), ((\alpha_{ii} - \alpha_{ij})^+, \min(M_i, N_j))) \log \text{SNR} + o(\log \text{SNR}) \\
&= [\alpha_{ii} \min((M_i - N_j)^+, N_i) + (\alpha_{ii} - \alpha_{ij})^+ \min((N_i - (M_i - N_j)^+)^+, N_j, M_i)] \log \text{SNR} \\
&\quad + o(\log \text{SNR}) \\
&\stackrel{(c)}{=} [\alpha_{ii} \min((M_i - N_j)^+, N_i) + (\alpha_{ii} - \alpha_{ij})^+ (\min(M_i, N_i) - \min((M_i - N_j)^+, N_i))] \log \text{SNR} \\
&\quad + o(\log \text{SNR}), \quad (154)
\end{aligned}$$

where (a) follows from the fact that if $(\alpha_{ii} - \alpha_{ij})$ is less than zero we have

$$\text{SNR}^{(\alpha_{ii} - \alpha_{ij})^+} H_{ii} U_{ij1} (\Lambda_{ij})^{-1} U_{ij1}^\dagger H_{ii}^\dagger = o(\log \text{SNR}), \quad (155)$$

(b) follows from Lemma 15 and that $H_{ii} U_{ij1}$, $H_{ii} U_{ij1} \Lambda_{ij}^{-1/2}$ and $H_{ii} [U_{ij2} \ U_{ij1} \Lambda_{ij}^{-1/2}]$ are all full rank with

probability 1; (c) follows from some simple manipulations. ■

The rest of the section considers the 6 terms in $\mathcal{R}_o(0)$ in (13)-(18), and finds the GDoF region for the MIMO IC with feedback.

First term: According to the first bound in $\mathcal{R}_o(0)$, we have

$$\begin{aligned} & \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}H_{21}^\dagger) \\ \stackrel{(a)}{=} & f((N_1, (\alpha_{11}, M_1), (\alpha_{21}, M_2))) \log \text{SNR} + o(\log \log \text{SNR}), \end{aligned} \quad (156)$$

where (a) is obtained from (15). Now, dividing both sides by $\log \text{SNR}$, we get the first GDoF expression.

Second term: The second bound is similar to the first bound by replacing 1 and 2 in the indices.

Third term: According to the third bound in $\mathcal{R}_o(0)$, we have

$$\begin{aligned} & \log \det(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger) + \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \\ & \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^\dagger(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}\sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^\dagger) \\ \stackrel{(a)}{=} & \alpha_{12} \min(M_1, N_2) + \alpha_{11} \min((M_1 - N_2)^+, N_1) + \\ & (\alpha_{11} - \alpha_{12})^+ \min(M_1, N_1) - \min((M_1 - N_2)^+, N_1) + o(\log \text{SNR}), \end{aligned} \quad (157)$$

where (a) is obtained from Lemma 13 and Lemma 16. Now, dividing both sides by $\log \text{SNR}$, the third GDoF bound results.

Fourth term: The fourth term is similar to the third term by replacing 1 and 2 in the indices.

Fifth term: According to the fifth bound in $\mathcal{R}_o(0)$, we have

$$\begin{aligned} & \log \det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger) \\ & + \log \det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^\dagger(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}\sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^\dagger) \\ \stackrel{(a)}{=} & f((N_2, (\alpha_{22}, M_2), (\alpha_{12}, M_1))) + \alpha_{11} \min((M_1 - N_2)^+, N_1) + \\ & (\alpha_{11} - \alpha_{12})^+ (\min(M_1, N_1) - \min((M_1 - N_2)^+, N_1)) + o(\log \text{SNR}), \end{aligned} \quad (158)$$

where (a) is obtained from Lemma 15 and Lemma 16. Now, dividing both sides by $\log \text{SNR}$, the fifth GDoF bound results.

Sixth term: The sixth term is similar to the fifth term by replacing 1 and 2 in the indices.

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